September 2013

QR Exam in Numerical Analysis

1. Euler's method is used to solve the IVP $U' = AU$, $U(0) = U_0$ where A is chosen from

$$A_1 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In each case determine which values of step size k ensure that the numerical solution u^n remains bounded as $n \to \infty$ for all initial vectors U_0

2. Consider the following scheme for $du/dt = f(u)$

$$\frac{1.5u^{n+1} - 2u^n + 0.5u^{n-1}}{\Delta t} = f(u^{n+1})$$

(a) Show that the scheme is stable.

(b) Show that the region of absolute stability contains the negative real axis.

(c) Consider the difference scheme

$$\frac{1.5u^{n+1} - 2u^n + 0.5u^{n-1}}{\Delta t} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}, \quad j = 1, ..., N-1, n = 0, 1, ...$$

for the heat equation $u_t = u_{xx}$, with boundary conditions

$$u_0^n = u_N^n = 0, \quad n = 0, 1, ...$$

Does the result in part b tell you anything about the stability of this discretization?

3. Consider the linear advection equation $u_t + au_x = 0$, aproximated by the implicit scheme

$$\frac{u_j^{n+1} - u_j^n}{k} + aD_0\frac{u_j^{n+1} + u_j^n}{2} = 0.$$

(a) Use the Fourier method to analyze the L_2 stability of the scheme.

(b) Rederive the result in part I using the energy method.

4. The 2-point BVP $u'' = f(u)$, $u(0) = u'(1) = 0$ is solved using the centered difference scheme with mesh size h. Show that the method is L_2-stable.

5. Consider the difference scheme

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) + kD_+D_-u_j^n.$$

(a) For which values of $r = k/h^2$ is the scheme consistent with the heat equation $u_t = u_{xx}$?

(b) For which values of $r = k/h^2$ is the scheme stable in the ∞-norm?

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1. True or false? Give a brief reason to justify your answer.

a) If $u_t - cu_x = 0$ and $u(x,0) \geq 0$ for all x, then $u(x,t) \geq 0$ for all x and all $t > 0$.

b) If $u_{xx} + u_{yy} = 0$ on a domain D and $m \leq u \leq M$ on ∂D, then $m \leq u \leq M$ on D, where m and M are constants.

c) The solution space of the heat equation $u_t = u_{xx}$ is a two dimensional vector space.

d) $L_1(\mathbb{R})$ is a proper subspace of $L_1^{loc}(\mathbb{R})$

e) If $f_n(x) = ne^{-nx}$, then $f_n \to 0$ in $L_2(0,1)$ as $n \to \infty$.

2. Let $f(x) = e^{-a|x|}$, where $a > 0$.

a) Find the Fourier transform of f.

b) Apply the Plancherel theorem and the result in (a) to evaluate $\int_{-\infty}^{\infty} \frac{1}{(t^2+a^2)(t^2+b^2)}\, dt$.

3. Consider the Sturm-Liouville problem $u'' + \lambda u = 0, \quad u(0) = u'(1) = 0$.

a) Find the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the problem.

b) Find the Green's function for the problem.

c) Write down the expansion of the Green's function in terms of the eigenvalues and eigenfunctions.

4. Consider the heat equation with a source term, $u_t = u_{xx} + \cos x$, for $0 < x < \pi$, with initial condition

$$u(x,0) = \begin{cases} 1 \ , & 0 < x < \frac{\pi}{2} \\ -1 \ , & \frac{\pi}{2} < x < \pi \end{cases}$$

and boundary conditions $u(0,t) = 1\,;\ u(\pi,t) = -1$.

a) Find $u(x,t)$ in the form of a Fourier cosine series.

b) Prove that $u(x,t)$ is continuous for any $t > 0$.

c) Find $\lim_{t\to\infty} u(x,t)$.

5. Define $u(x,y) = \frac{1}{2\pi} \log \sqrt{x^2+y^2}$.

a) Show that $\Delta u = 0$ for $(x,y) \neq (0,0)$.

b) Show that u is a fundamental solution of the Laplace equation on $\mathbb{R}^2$.