1. Consider the operator $Lf = x^2 \frac{d^2 f}{dx^2} + 2x \frac{df}{dx} + \frac{1}{4} f$, $1 < x < 2$, $f(1) = 0 = f(2)$.

(a) Show that this is a regular Sturm-Liouville problem, i.e., that the differential operator together with the boundary conditions are self-adjoint.

(b) Find all the eigenvalues λ_n and *normalized* eigenfunctions $\{\phi_n(x)\}_{n=1}^{\infty}$ satisfying $(L + \lambda_n)\, \phi_n = 0$, $\phi_n(1) = 0 = \phi_n(2)$, and $\|\phi_n\| = 1$.

(Hint: Look for linear combinations of functions like x^p. Also, $\int_0^{\pi} \sin^2 n\theta \, d\theta = \frac{\pi}{2}$ for all integers n.)

2. Let $-\infty < a < b < \infty$ and consider functions $f_n, f, g \in L^2(a,b)$.

(a) Show that if $f_n \to f$ in norm, then $\langle f_n, g\rangle \to \langle f, g\rangle$ for all g.

(Hint: Apply the Cauchy-Schwarz inequality to $\langle f_n - f, g\rangle$.)

(b) Show that $\big| \|f\| - \|g\| \big| \leq \|f-g\|$.

(c) Using (b), deduce that if $f_n \to f$ in norm, then $\|f_n\| \to \|f\|$.

3. Recall that if all the integrals make sense, then $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \, e^{i\xi x} \left(\int_{-\infty}^{\infty} dy \, e^{-i\xi y} f(y) \right)$.

(a) What is the Fourier transform of $f(x) = e^{-a|x|}$? (a is real and positive)

(b) What is the inverse Fourier transform of $\frac{e^{-ib\xi}}{\xi^2 + \mu^2}$? ($b$ is real, as is $\mu > 0$)

4. Recall that a sequence of distributions $\{F_n\}_{n=1}^{\infty}$ in $D'(R)$ converges weakly to $F \in D'(R)$ if for any test function $\varphi(x) \in C_0^{(\infty)}(R)$, $F_n[\varphi] \to F[\varphi]$.

(a) Suppose F_n is the distribution given by the function

$$F_n(x) = \begin{cases} n \text{ for } 0 \leq x \leq \frac{1}{n} \\ 0 \text{ if } x < 0 \text{ or } x > \frac{1}{n} \end{cases}$$

Prove that $F_n \to \delta$ weakly as $n \to \infty$. Remember that $\delta[\varphi] = \varphi(0)$.

(Hints: In one approach, note that for each $y \in [0,1]$, $\varphi(\frac{y}{n}) \to \varphi(0)$ as $n \to \infty$. Also, φ is bounded on $[0,1]$ and the Dominated Convergence Theorem may be useful. In another approach, expand $\varphi(x)$ in a Taylor series about $x = 0$, keeping track of the size of the remainder. Remember that φ' is bounded on $[0,1]$.)

(b) Let $H(x)$ be the step function, $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geq 0$. Prove that $H' = \delta$.

5. Let $L = -\frac{d^2}{dx^2} + m^2$ (with m real and positive) acting on functions which vanish as $|x| \to \infty$.

The Green's function for L satisfies $\left\{-\frac{d^2}{dx^2} + m^2\right\} G(x,y) = \delta(x-y)$ with $G(x,y) \to 0$ as $|x| \to \infty$.

(a) What is $G(x,y)$?

(Hint: Find solutions of the homogeneous problem that vanish as $x \to -\infty$, or as $x \to +\infty$.)

(b) What is the Fourier transform (in x; y is just a parameter) of $G(x,y)$?

(Hint: Take the Fourier transform of the differential equation defining the Green's function.)

Applied Analysis: Numerical Analysis

1. **Polynomial Interpolation** Define the nodes $x_0 = 1$, $x_1 = 2$, and $x_2 = 4$. Consider the function $f(x) = 2^x$.

   (a) Write down the Lagrange polynomial $\ell_0(x)$ associated with the node x_0.

   (b) Find the Newton interpolant polynomial $p(x)$ of f for the given nodes x_0, x_1, x_2.

   (c) Find a rigorous upper bound on the error between f and its Newton interpolant p at the point $x = 3$. For reference, the interpolation error theorem is stated below. You should evaluate explicitly as much of the error bound as you can.

   Theorem 1. *Suppose that f has $n+1$ continuous derivatives on $[a, b]$, and $x_0, \ldots, x_n$ are distinct points in $[a, b]$. For any $x \in [a, b]$, there is some $\xi(x) \in [a, b]$ such that the Newton interpolant $p(x)$ satisfies*

   $$f(x) - p(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n).$$

2. **Iterative Methods** Consider the system of linear equations,

   $$\begin{array}{rcrcr} 2x & + & y & = & 1 \\ x & + & 2y & = & -1 \end{array}$$

   Starting from the initial iterate $(x, y) = (0, 0)$, perform one step of

   (a) Jacobi's Method

   (b) Gauss–Seidel Method

   (c) SOR with parameter $\omega = \frac{3}{2}$

3. **Condition number and SVD**
Let

$$A = \begin{bmatrix} I & B \\ B^* & I \end{bmatrix}$$

where B is an $m \times m$ matrix with $\|B\|_2 < 1$.

(a) What are the eigenvectors of A? **Hint:** Consider the SVD of the matrix B.

(b) Show that the condition number of A is

$$\kappa_2(A) = \frac{1 + \|B\|_2}{1 - \|B\|_2}.$$

4. **Matrix norms**
Consider a norm defined for a matrix $A \in \mathbb{C}^{m \times n}$ by

$$\|A\| = \max_{i,j}(|A_{i,j}|).$$

(a) Is this a valid norm? Prove your statement.

(b) Does this norm satisfy the bound on matrix products $\|AB\| \leq \|A\|\|B\|$? Prove your statement or provide a (simple) counter example.

5. **QR Algorithm**
Let A be a Hermitian matrix which we reduce to T, a symmetric tridiagonal matrix, in preparation for input to the QR Algorithm. This preliminary reduction to tridiagonal form would be of little use if the steps of the QR algorithm did not preserve this structure. Fortunately, they do.

(a) In the QR factorization $T = QR$ of the symmetric tridiagonal matrix T, which entries of R are in general nonzero? Which entries of Q? (Although, in practice, we do not form Q explicitly.)

(b) Show that the symmetric tridiagonal structure is recovered when the product RQ is formed; i.e., that RQ is also a symmetric tridiagonal matrix.