Department of Mathematics, University of Michigan
Real Analysis Qualifying Exam
January 09, 2024; Morning Session

Problem 1: Let $n \in \mathbb{N}$ and n points $x_1, \ldots, x_n \in [0,1)$ be fixed. Assume that a Lebesgue measurable set $E \subset [0,1)$ satisfies $m(E) > 1 - 3/n$. Prove that there is $x \in [0,1)$ such that $\{x - x_k\} \in E$ for all indices $k = 1, \ldots, n$ *except at most two.* (Above, $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of x.)

Solution: Denote $E_k := \{\{x + x_k\},\ x \in E\}$, the 'periodic shift' of E by x_k. Writing $E = (E \cap [0, 1-x_k)) \cup (E \cap [1-x_k, 1))$ it is easy to see that $m(E_k) = m(E)$. Clearly, $\{x - x_k\} \in E$ if and only if $x \in E_k$. Let $\mathbb{1}_{E_k}$ be the characteristic function of E_k. If $x \in [0,1)$ does not belong to at least three of the sets E_k, then $\sum_{k=1}^n \mathbb{1}_{E_k}(x) \le n-3$ and if this happened for all $x \in [0,1)$, then we would have

$$\int_0^1 \sum_{k=1}^n \mathbb{1}_{E_k}(x)dx \ = \ \sum_{k=1}^n m(E_k) \le n-3,$$

which is a contradiction as $m(E_k) = m(E) > 1 - \frac{3}{n}$ for all $k = 1, \ldots, n$.

Problem 2: Let two functions $f, g : [0,1] \to [0,1]$ be defined as $f(x) := x^2 |\sin \frac{1}{x}|$ (and $f(0) := 0$) and $g(x) := \sqrt{x}$. Which of the four functions f, g, $f \circ g$, and $g \circ f$ are *absolutely* continuous on $[0,1]$?

Solution: Let $f_1(x) := x^2 \sin(\frac{1}{x})$ (and $f_1(0) := 0$). The function f_1 is differentiable on $(0,1]$ and its derivative $f_1'(x) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$ is bounded. Therefore, f_1 is absolutely continuous and so is the function $f = |f_1|$.

The function g is also differentiable on $(0,1]$ and its derivative $g'(x) = \frac{1}{2\sqrt{x}}$ belongs to $L^1([0,1])$. Therefore, g is absolutely continuous too.

The composition $f \circ g$ is absolutely continuous since both f, g are absolutely continuous and g is *monotone*. Indeed, as f is absolute continuous, for each $\varepsilon > 0$ one can find $\delta = \delta(\varepsilon) > 0$ such that $\sum_{k=1}^n |f(v_k) - f(u_k)| < \varepsilon$ for each collection of pairwise disjoint segments $[u_k, v_k] \subset [0,1]$ with $\sum_{k=1}^n (v_k - u_k) < \delta$. Then, since g is absolutely continuous, for each $\delta > 0$ one can find $\rho = \rho(\delta) > 0$ such that $\sum_{k=1}^n |g(b_k) - g(a_k)| < \delta$ for each collection of pairwise disjoint segments $[a_k, b_k] \subset [0,1]$ with $\sum_{k=1}^n (b_k - a_k) < \rho$. Since g is monotone, the segments $[u_k, v_k] := [g(a_k), g(b_k)]$ are also disjoint, which completes the argument.

However, the composition $(g \circ f)(x) = x \cdot \sqrt{|\sin \frac{1}{x}|}$ is *not* absolutely continuous. To see this, note that this function vanishes at points $a_k = (\pi k)^{-1}$, $k \in \mathbb{N}$ and $(g \circ f)(b_k) = b_k$ at points $b_k = (\pi(k - \frac{1}{2}))^{-1}$, $k \in \mathbb{N}$. As the series $\sum_{k=1}^\infty b_k$ diverges, the function $g \circ f$ is not of bounded variation and hence not absolutely continuous.

Problem 3: **(a)** Let $f \in L^p([0,1])$ for some $p > 3$. Prove that the function $F(x) := \int_0^x (x-t)^{-\frac{2}{3}} f(t)dt$ is well-defined and bounded on the segment $x \in [0,1]$. Does this statement remain true for $p = 3$?

(b) Assume now that $p > \frac{3}{2}$. Prove that the integral $F(x) := \int_0^x (x-t)^{-\frac{2}{3}} f(t)dt$ converges for almost every $x \in [0,1]$ and that $F \in L^2([0,1])$.

Solution: **(a)** Let $p > 3$. We can use Hölder's inequality to write

$$|F(x)| \ \le \ \left(\int_0^x (x-t)^{-\frac{2}{3} \cdot \frac{p}{p-1}} dt \right)^{\frac{p-1}{p}} \left(\int_0^x |f(t)|^p dt \right)^{\frac{1}{p}}.$$

The second factor is bounded by $\|f\|_p$ and the first one admits (since $\frac{2}{3}\cdot\frac{p}{p-1}<1$) a uniform estimate

$$\left(\int_0^x (x-t)^{-\frac{2p}{3(p-1)}}\,dt\right)^{\frac{p}{p-1}} \;\le\; \left(\int_0^\infty y^{-\frac{2p}{3(p-1)}}\,dy\right)^{\frac{p}{p-1}} \;=\; \left(1-\frac{2p}{3(p-1)}\right)^{-\frac{p}{p-1}}.$$

If $p=3$, then the integral defining $F(x)$ does *not* necessarily converge pointwise. For instance, given a point $x_0\in(0,1)$ one can consider a function

$$f(x) := \frac{1}{|x-x_0|^{\frac{1}{3}}\log|x-x_0|}\,, \quad x\in[0,1].$$

The function $x\mapsto |x-x_0|^{-1}(\log|x-x_0|)^{-3}$ has an *integrable* singularity at x_0 (the antiderivative of this function equals $\mathrm{sign}(x-x_0)\cdot(\log|x-x_0|)^{-2}$), this is why $f\in L^3([0,1])$. However, the integral

$$F(x_0) \;=\; \textstyle\int_0^{x_0}(x_0-t)^{-\frac{2}{3}}f(t)dt \;=\; \int_0^{x_0}(x_0-t)^{-1}(\log(x_0-t))^{-1}dt$$

diverges (the antiderivative equals $-\log|\log(x_0-t)|$).

(b). Without loss of generality one can assume that $f\ge 0$: replace f by $|f|$ otherwise. (In particular, if we are able to prove that $F(x)<+\infty$ for almost every x with f replaced by $|f|$, this means that the Lebesgue integral defining $F(x)$ for the original function f converges for almost every x as well.) The key observation is that one can write

$$\begin{aligned}\int_0^1 (F(x))^2dx \;&=\; \int_0^1\left(\int_0^x (x-t)^{-\frac{2}{3}}f(t)dt\right)^2 dx\\ &=\; \int_0^1\left(\int_0^x\int_0^x (x-t)^{-\frac{2}{3}}(x-s)^{-\frac{2}{3}}f(t)f(s)dtds\right)dx\\ &=\; \int_0^x\int_0^x\left(\int_{\max\{s,t\}}^1 (x-t)^{-\frac{2}{3}}(x-s)^{-\frac{2}{3}}dx\right)f(t)f(s)dtds\,.\end{aligned}$$

(Note that we can use Tonelli's theorem even if we do not know that $F(x)<+\infty$ almost everywhere.) The inner integral can be estimated as follows:

$$\int_{\max\{s,t\}}^1 (x-t)^{-\frac{2}{3}}(x-s)^{-\frac{2}{3}}dx \;\le\; \int_0^\infty y^{-\frac{2}{3}}(y+|t-s|)^{-\frac{2}{3}}dy \;=\; C\cdot|t-s|^{-\frac{1}{3}},$$

where $C=\int_0^\infty y^{-\frac{2}{3}}(y+1)^{-\frac{2}{3}}dy<\infty$. Therefore,

$$\begin{aligned}\int_0^1 (F(x))^2dx \;&\le\; C\cdot\int_0^1\int_0^1 |t-s|^{-\frac{1}{3}}f(t)f(s)dtds\\ &\le\; C\cdot\left(\int_0^1\int_0^1 |t-s|^{-\frac{2}{3}}dtds\right)^{\frac{1}{2}}\cdot\|f\|_2\,,\end{aligned}$$

where we used the Cauchy–Schwarz inequality and $\int_0^1\int_0^1(f(t))^2(f(s))^2dtds=\|f\|_2^2$ in the second line. Finally, it is easy to see (e.g., by changing the variables t,s to $t-s$ and $t+s$) that $\int_0^1\int_0^1|t-s|^{-\frac{2}{3}}dtds<+\infty$, which completes the proof. In particular, $F(x)<+\infty$ for almost every $x\in[0,1]$ since $\int_0^1(F(x))^2dx<+\infty$.

Problem 4: Let $f_n:[0,1]\to\mathbb{R}$ be a sequence of measurable functions. Assume that $f_n(x)\to 0$ for almost every $x\in[0,1]$. Prove that one can find a sequence of real numbers C_n such that $C_n\to+\infty$ and $C_nf_n(x)\to 0$ for almost every $x\in[0,1]$.

Solution: Without loss of generality, one can assume that all functions f_n are non-negative and that the sequence $f_n(x)$ is *monotone decreasing* for each x: replace f_n by $\max_{m\geq n}|f_m(x)|$ otherwise. (Note that the maximum is attained since $f_n(x)\to 0$ as $n\to\infty$ and that the supremum of measurable functions taken over a countable set is again a measurable function.)

For each $\varepsilon>0$ the sets $E_n := \{x\in[0,1] : f_n(x)\geq\varepsilon\}$ are decreasing (i.e., $E_n\supset E_{n+1}$) and $m(\bigcap_{n=1}^{\infty}E_n)=0$ since $f_n(x)\to 0$ almost everywhere. Therefore (continuity of the measure, note that $m([0,1])<\infty$), for each $\varepsilon>0$ and each $\rho>0$ there exists $n=n(\varepsilon,\rho)\in\mathbb{N}$ such that

$$m(\{x\in[0,1] : f_n(x)\geq\varepsilon\}) \;\leq\; \rho.$$

In particular, one can find an increasing sequence of indices $n_1<n_2<\ldots$ such that

$$m(\{x\in[0,1] : f_{n_k}(x)>(k+1)^{-2}\}) \;\leq\; 2^{-k}.$$

(The only important property of this choice of parameters is the summability of the series $\sum_{k=1}^{\infty}2^{-k}$.) Let us now define $n_0:=0$ and

$$C_n := k \quad \text{if} \;\; n_{k-1}\leq n<n_k\,.$$

By construction,

$$m(\{x\in[0,1] : C_{n_k}f_{n_k}(x)>(k+1)^{-1}\}) \;\leq\; 2^{-k}$$

and hence (Borel–Cantelli's lemma) almost every point $x\in[0,1]$ belongs only to *finitely many* such sets. This implies that $\lim_{k\to\infty}C_{n_k}f_{n_k}(x)=0$ and hence $\lim_{n\to\infty}C_nf_n(x)=0$ for almost every $x\in[0,1]$.

Problem 5: Let $n\geq 2$. For $f\in L^1(\mathbb{R}^n)$ and $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, denote

$$(M_{\text{cube}}f)(x) := \sup_{a>0}\frac{1}{(2a)^n}\int_{[x_1-a,x_1+a]\times\ldots[x_n-a,x_n+a]}|f(y)|dy,$$
$$(M_{\text{rect}}f)(x) := \sup_{a_1,\ldots,a_n>0}\frac{1}{2^na_1\ldots a_n}\int_{[x_1-a_1,x_1+a_1]\times\ldots\times[x_n-a_n,x_n+a_n]}|f(y)|dy$$

(a) Use the Hardy–Littlewood maximal inequality to prove that there exists a constant $C_{\text{cube}}>0$ (depending only on n) such that $m_n(\{x : (M_{\text{cube}}f)(x)>\lambda\})\leq C_{\text{cube}}\cdot\lambda^{-1}\|f\|_1$ for all $\lambda>0$ and $f\in L^1(\mathbb{R}^n)$.

(b) Prove that a similar fact for $M_{\text{rect}}f$ does *not* hold: there is *no* constant $C_{\text{rect}}>0$ such that $m_n(\{x : (M_{\text{rect}}f)(x)>\lambda\})\leq C_{\text{rect}}\cdot\lambda^{-1}\|f\|_1$ for all $\lambda>0$ and $f\in L^1(\mathbb{R}^n)$. [*Hint:* it may be easier to work out the case $n=2$ first.]

Solution: (a) Since $[x_1-a,x_1+a]\times\ldots\times[x_n-a,x_n+a]\;\subset B(x,2^{\frac{n}{2}}a)$, one has

$$\frac{1}{(2a)^n}\int_{[x_1-a,x_1+a]\times\ldots[x_n-a,x_n+a]}|f(y)|dy \;\leq\; b_n\cdot\frac{1}{m_n(B(x,2^{\frac{n}{2}}a))}\int_{B(x,2^{\frac{n}{2}}a)}|f(y)|dy,$$

where $b_n := m_n(B(x,2^{\frac{n}{2}}a))/(2a)^n$ is a constant depending on n only. (There is no dependence on a since the numerator and the denominator scale in the same way.) Therefore,

$$m_n(\{x : (M_{\text{cube}}f)(x)>\lambda\}) \;\leq\; m_n(\{x : (Mf)(x)>b_n^{-1}\lambda\}) \;\leq\; Cb_n\cdot\lambda^{-1}\cdot\|f\|_1,$$

where C is the constant from the usual Hardy–Littlewood maximal inequality.

(b) For simplicity, let us first assume that $n = 2$. Consider $f(x) = \mathbb{1}_{[-1,1]^2}(x)$. Clearly, $\|f\|_1 = 4$ and

$$(M_{\text{rect}} f)(x) \;\geq\; \frac{1}{(|x_1|+1)(|x_2|+1)} \quad \text{for all} \;\; x = (x_1, x_2) \in \mathbb{R}^2$$

as one can take $a_1 := |x_1| + 1$ and $a_2 := |x_2| + 1$ in the definition of M_{rect}. Hence, for all $\lambda < 1$ we have

$$\begin{aligned} m_2(\{x : (M_{\text{rect}} f)(x) > \lambda\}) \;&\geq\; m_2(\{x = (x_1, x_2) : (|x_1|+1)(|x_2|+1) < \lambda^{-1}\}) \\ &=\; 4m_2((x_1, x_2) \in \mathbb{R}^2_+ : (x_1+1)(x_2+1) < \lambda^{-1}) \\ &=\; 4\int_0^{\lambda^{-1}-1} \left(\frac{\lambda^{-1}}{x_1+1} - 1\right) dx_1 \\ &=\; 4(\lambda^{-1}|\log\lambda| - \lambda^{-1} + 1) \end{aligned}$$

If $\lambda \to 0$, this lower bound clearly contradicts to the hypothetical uniform estimate $m_2(\{x : (M_{\text{rect}} f)(x) > \lambda\}) \leq C_{\text{rect}} \cdot 4\lambda^{-1}$ from above.

Similar arguments apply if $n > 2$ and $f(x) = \mathbb{1}_{[-1,1]^n}(x)$. For instance, one can use a crude estimate

$$\begin{aligned} & m_n(\{x \in \mathbb{R}^n_+ : \; (x_1+1)\ldots(x_n+1) < \lambda^{-1}\}) \\ & \qquad\qquad \geq\; m_2(\{(x_1, x_2) \in \mathbb{R}^2_+ : (x_1+1)(x_2+1) < 2^{-(n-2)}\lambda^{-1}\}), \end{aligned}$$

which is obtained by additionally requiring that $x_3, \ldots, x_n \in [0, 1]$.