Department of Mathematics, University of Michigan
Real Analysis Qualifying Exam
January 09, 2024; Morning Session

Problem 1: Let $n \in \mathbb{N}$ and n points $x_1, \ldots, x_n \in [0,1)$ be fixed. Assume that a Lebesgue measurable set $E \subset [0,1)$ satisfies $m(E) > 1 - 3/n$. Prove that there is $x \in [0,1)$ such that $\{x - x_k\} \in E$ for all indices $k = 1, \ldots, n$ *except at most two.* (Above, $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of x.)

Problem 2: Let two functions $f, g : [0,1] \to [0,1]$ be defined as $f(x) := x^2 |\sin \frac{1}{x}|$ (and $f(0) := 0$) and $g(x) := \sqrt{x}$. Which of the four functions f, g, $f \circ g$, and $g \circ f$ are *absolutely* continuous on $[0,1]$?

Problem 3: **(a)** Let $f \in L^p([0,1])$ for some $p > 3$. Prove that the function $F(x) := \int_0^x (x-t)^{-\frac{2}{3}} f(t)dt$ is well-defined and bounded on the segment $x \in [0,1]$. Does this statement remain true for $p = 3$?

(b) Assume now that $p > \frac{3}{2}$. Prove that the integral $F(x) := \int_0^x (x-t)^{-\frac{2}{3}} f(t)dt$ converges for almost every $x \in [0,1]$ and that $F \in L^2([0,1])$.

Problem 4: Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of measurable functions. Assume that $f_n(x) \to 0$ for almost every $x \in [0,1]$. Prove that one can find a sequence of real numbers C_n such that $C_n \to +\infty$ and $C_n f_n(x) \to 0$ for almost every $x \in [0,1]$.

Problem 5: Let $n \geq 2$. For $f \in L^1(\mathbb{R}^n)$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, denote

$$(M_{\text{cube}} f)(x) := \sup_{a>0} \frac{1}{(2a)^n} \int_{[x_1-a, x_1+a] \times \ldots [x_n - a, x_n + a]} |f(y)| dy,$$

$$(M_{\text{rect}} f)(x) := \sup_{a_1, \ldots, a_n > 0} \frac{1}{2^n a_1 \ldots a_n} \int_{[x_1-a_1, x_1+a_1] \times \ldots \times [x_n - a_n, x_n + a_n]} |f(y)| dy$$

(a) Use the Hardy–Littlewood maximal inequality to prove that there exists a constant $C_{\text{cube}} > 0$ (depending only on n) such that $m_n(\{x : (M_{\text{cube}} f)(x) > \lambda\}) \leq C_{\text{cube}} \cdot \lambda^{-1} \|f\|_1$ for all $\lambda > 0$ and $f \in L^1(\mathbb{R}^n)$.

(b) Prove that a similar fact for $M_{\text{rect}} f$ does *not* hold: there is *no* constant $C_{\text{rect}} > 0$ such that $m_n(\{x : (M_{\text{rect}} f)(x) > \lambda\}) \leq C_{\text{rect}} \cdot \lambda^{-1} \|f\|_1$ for all $\lambda > 0$ and $f \in L^1(\mathbb{R}^n)$. [*Hint:* it may be easier to work out the case $n = 2$ first.]