Notation: \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \)

(1) Find all solutions of \( \cos z = 1 + 100z^2 \) in the unit disk \( |z| < 1 \).

**SOLUTION:** Using Rouché’s Theorem with \( 100z^2 \) as the dominant term we see that \( f(z) = 1 + 100z^2 - \cos z \) has two zeros in the unit disk, which are entirely accounted for by the double zero at the origin.

(2) Find

\[
\sup \{ |f(1)| : f \text{ is holomorphic on } \mathbb{C} \setminus \{0\} \text{ and satisfies } |f(z)| \leq 7|z|^{-3/2} \}.
\]

**SOLUTION:** A function \( f \) satisfying the conditions must have a removable singularity at infinity and at most a simple pole at the origin, hence \( f \) must be rational. Since the extended \( f \) must have at least a double zero at infinity and there is no room to balance these with two poles, \( f \) must be identically zero. Hence the desired sup is zero.

(3) Let \( f_k : \mathbb{D} \to \mathbb{C} \) be a sequence of holomorphic functions forming a normal family (that is to say, every subsequence of \( (f_k) \) has a further subsequence convering uniformly on each compact subset of \( \mathbb{D} \)). Further, let \( h_k : \mathbb{D} \to \mathbb{D} \) be holomorphic functions satisfying \( h_k(0) = 0 \). Prove that the functions

\[
g_k(z) = f_k(h_k(z))
\]

form a normal family.

**SOLUTION:** The normality condition on the \( f_k \) is equivalent to the condition that for \( 0 < r < 1 \) there is \( M_r \) so that \( |f_k(z)| \leq M_r \) when \( |z| \leq r \). By Schwarz’s Lemma \( |h_k(z)| \leq r \) when \( |z| \leq r \). Combining these facts we have \( |g_k(z)| \leq M_r \) when \( |z| \leq r \). Thus the \( g_k \) form a normal family.

(4) Let \( D_1, D_2 \subset \mathbb{C} \) be disks with the property that the circles \( \text{Bd } D_1, \text{Bd } D_2 \) intersect in exactly two points. Under what additional hypothesis will there exist a bijective rational map from \( D_1 \cap D_2 \) to \( \mathbb{D} \)?

**SOLUTION:** The interior angle of intersection must be of the form \( \pi/n \) for some natural \( n \) so that it can be converted to \( \pi \) by a rational function with derivative vanishing to order \( n \) at the intersection points \( p_1, p_2 \).
If this condition holds then the desired map can be constructed by composing maps of the form

\[ z_1 = e^{i\theta} \frac{z - p_1}{z - p_2} \quad \text{[mapping to sector with } \mathbb{R}_+ \text{ as bottom edge]} \]

\[ z_2 = z_1^n \]

\[ w = \frac{z_2 - i}{z_2 + i}. \]

(5) Suppose that \( f \) is holomorphic on \( \{ z \in \mathbb{C} : |z| > r \} \) for some \( r < 1 \). Suppose further that \( zf(z) \to 1 \) as \( z \to \infty \).

(a) Evaluate \( \int_{|z|=1} zf'(z) \, dz \).

(b) Show that \( \int_{|z|=1} |f'(z)||dz| \geq 2\pi \).

(c) When does equality hold in (b)?

**SOLUTION:**

(a) We have \( f(z) = \frac{1}{z} + a_0 + a_1 z + \ldots \) and \( zf'(z) = -\frac{1}{z} + a_1 z + \ldots \). Thus

\[ \int_{|z|=1} zf'(z) \, dz = -2\pi i. \]

(b) This follows from

\[ 2\pi = \left| \int_{|z|=1} zf'(z) \, dz \right| \leq \int_{|z|=1} |f'(z)| \, |dz|. \]

Remark: Note that \( \int_{|z|=1} |f'(z)||dz| \) is the length of the image of the unit circle.

(c) From (a) we have

\[ 2\pi = \int_0^{2\pi} \text{Re} \left( e^{it} f'(e^{it}) \cdot e^{it} \right) \, dt \leq \int_{|z|=1} |f'(z)| \, |dz| \]

with equality if and only if \( e^{it} f'(e^{it}) \cdot e^{it} \) is non-negative.

By Schwarz reflection, \( \phi(z) \equiv z^2 f'(z) \) extends to a holomorphic function on \( \mathbb{C} \) satisfying \( \phi(z) = \phi(1/z) \). Thus \( \phi(0) = -1 = \phi(\infty) \) and in fact \( \phi(z) = -1 \) for all \( z \). Thus \( f'(z) = -\frac{1}{z^2} \) and \( f(z) = \frac{1}{z} + C \); original limit condition requires \( C = 0 \).