Qualifying Review Exam
Complex Analysis
January 2024

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

(1) Find all solutions of $\cos z = 1 + 100z^2$ in the unit disk $|z| < 1$.

SOLUTION: Using Rouché's Theorem with $100z^2$ as the dominant term we see that $f(z) = 1 + 100z^2 - \cos z$ has two zeros in the unit disk, which are entirely accounted for by the double zero at the origin.

(2) Find

$$\sup \left\{ |f(1)| : f \text{ is holomorphic on } \mathbb{C} \setminus \{0\} \text{ and satisfies } |f(z)| \leq 7|z|^{-3/2} \right\}.$$

SOLUTION: A function f satisfying the conditions must have a removable singularity at infinity and at most a simple pole at the origin, hence f must be rational. Since the extended f must have at least a double zero at infinity and there is no room to balance these with two poles, f must be identically zero. Hence the desired sup is zero.

(3) Let $f_k : \mathbb{D} \to \mathbb{C}$ be a sequence of holomorphic functions forming a normal family (that is to say, every subsequence of (f_k) has a further subsequence convering uniformly on each compact subset of $\mathbb{D}$). Further, let $h_k : \mathbb{D} \to \mathbb{D}$ be holomorphic functions satisfying $h_k(0) = 0$. Prove that the functions

$$g_k(z) = f_k\left(h_k(z)\right)$$

form a normal family.

SOLUTION: The normality condition on the f_k is equivalent to the condition that for $0 < r < 1$ there is M_r so that $|f_k(z)| \leq M_r$ when $|z| \leq r$. By Schwarz's Lemma $|h_k(z)| \leq r$ when $|z| \leq r$. Combining these facts we have $|g_k(z)| \leq M_r$ when $|z| \leq r$. Thus the g_k form a normal family.

(4) Let $D_1, D_2 \subset \mathbb{C}$ be disks with the property that the circles Bd D_1, Bd D_2 intersect in exactly two points. Under what additional hypothesis will there exist a bijective *rational* map from $D_1 \cap D_2$ to $\mathbb{D}$?

SOLUTION: The interior angle of intersection must be of the form π/n for some natural n so that it can be converted to π by a rational function with derivative vanishing to order n at the intersection points p_1, p_2.

If this condition holds then the desired map can be constructed by composing maps of the form

$$z_1 = e^{i\theta}\frac{z-p_1}{z-p_2} \qquad \text{[mapping to sector with } \mathbb{R}_+ \text{ as bottom edge]}$$
$$z_2 = z_1^n$$
$$w = \frac{z_2 - i}{z_2 + i}.$$

(5) Suppose that f is holomorphic on $\{z \in \mathbb{C} : |z| > r\}$ for some $r < 1$. Suppose further that $zf(z) \to 1$ as $z \to \infty$.

(a) Evaluate $\int_{|z|=1} zf'(z)\,dz$.

(b) Show that $\int_{|z|=1} |f'(z)||dz| \geq 2\pi$.

(c) When does equality hold in (b)?

SOLUTION:

(a) *We have* $f(z) = \frac{1}{z} + a_0 + a_1 z + \ldots$ *and* $zf'(z) = -\frac{1}{z} + a_1 z + \ldots$. *Thus*

$$\int_{|z|=1} zf'(z)\,dz = -2\pi i.$$

(b) *This follows from*

$$2\pi = \left|\int_{|z|=1} zf'(z)\,dz\right| \leq \int_{|z|=1} \left|f'(z)\right| |dz|.$$

Remark: *Note that* $\int_{|z|=1} |f'(z)|\,|dz|$ *is the length of the image of the unit circle.*

(c) *From (a) we have*

$$2\pi = \int_0^{2\pi} \mathrm{Re}\left(e^{it} f'(e^{it}) \cdot e^{it}\right)\,dt \leq \int_{|z|=1} |f'(z)||dz|$$

with equality if and only if $e^{it}f'(e^{it}) \cdot e^{it}$ *is non-negative.*

By Schwarz reflection, $\phi(z) \stackrel{\text{def}}{=} z^2 f'(z)$ *extends to a holomorphic function on* $\mathbb{C}$ *satisfying* $\phi(z) = \overline{\phi(1/\overline{z})}$. *Thus* $\phi(0) = -1 = \phi(\infty)$ *and in fact* $\phi(z) = -1$ *for all z. Thus* $f'(z) = -\frac{1}{z^2}$ *and* $f(z) = \frac{1}{z} + C$*; original limit condition requires* $C = 0$.