Department of Mathematics, University of Michigan
Analysis Qualifying Exam, January 8, 2022
Solutions

Problem 1: Let E be a measurable subset of $[0,1]$. Suppose there exists $\alpha \in (0,1)$ such that

$$m(E \cap J) \ \geq \ \alpha \cdot m(J) \quad \text{for all subintervals } J \text{ of } [0,1] \ .$$

Prove that $m(E) = 1$.

Solution 1: Let $F = [0,1] \setminus E$. Then

$$m(F \cap J) \ \leq \ (1-\alpha) \cdot m(J) \quad \text{for all subintervals } J \text{ of } [0,1] \ .$$

Assume that $m(F) > 0$ and choose a cover of F by intervals $J_n, \ n \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} m(J_n) \leq (1+\alpha) m(F).$$

Then

$$m(F) = m\left(F \cap \left(\bigcup_{n=1}^{\infty} J_n\right)\right) \leq \sum_{n=1}^{\infty} m\left(F \cap J_n\right) \leq (1-\alpha) \sum_{n=1}^{\infty} m\left(J_n\right) \leq (1-\alpha^2) m(F).$$

This contradiction implies that $m(F) = 0$, and thus $m(E) = 1$.

Solution 2: By the Lebesgue differentiation theorem,

$$\frac{m(E \cap (a-\varepsilon, a+\varepsilon)}{2\varepsilon} \to \mathbb{1}_E(a) \quad \text{for almost all } a \in (0,1)$$

as $\varepsilon \to 0$. Since the assumption implies that

$$\liminf_{\varepsilon \to 0} \frac{m(E \cap (a-\varepsilon, a+\varepsilon)}{2\varepsilon} \geq \alpha \quad \text{for all } a \in (0,1),$$

$\mathbb{1}_E = 1$ a.e. on $[0,1]$ which means that $m(E) = 1$.

Problem 2: Let $f, g \in L^1(0,1)$. Assume for all functions $\phi \in C^\infty([0,1])$ with $\phi(0) = \phi(1)$ that

$$\int_0^1 f(t)\phi'(t) \ dt \ = \ -\int_0^1 g(t)\phi(t) \ dt \ .$$

Show that $f(\cdot)$ is absolutely continuous and $f' = g$.

Solution: Let $a, b \in (0,1), \ a < b$. Let $\varepsilon > 0$ be such that

$$\varepsilon < \min\left(\frac{a}{2}, \frac{b-a}{2}, \frac{1-b}{2}\right).$$

Define the function $\phi_\varepsilon : (0,1) \to \mathbb{R}$ by

$$\phi_\varepsilon(x) = \begin{cases} \frac{x-a+\varepsilon}{2\varepsilon}, & \text{if } x \in [a-\varepsilon, a+\varepsilon] \\ 1, & \text{if } x \in (a+\varepsilon, b-\varepsilon) \\ 1 - \frac{x-b+\varepsilon}{2\varepsilon}, & \text{if } x \in [b-\varepsilon, b+\varepsilon] \\ 0, & \text{otherwise.} \end{cases}$$

We can find a sequence of $C^\infty([0,1])$ functions ψ_n with $\psi_n(0) = \psi_n(1) = 0$ converging to ϕ_ε in the L^∞ norm. In addition to it, the functions ψ_n can be chosen so that

$$\psi_n' \to \frac{1}{2\varepsilon}\left(\mathbb{1}_{[a-\varepsilon,a+\varepsilon]} - \mathbb{1}_{[b-\varepsilon,b+\varepsilon]}\right) \text{ a.e.} \quad \text{and} \quad \|\psi_n'\|_\infty \le \frac{1}{\varepsilon}.$$

Applying the Lebesgue dominated convergence theorem, we obtain

$$\frac{1}{2\varepsilon}\left(\int_{a-\varepsilon}^{a+\varepsilon} f(t)\,dt - \int_{b-\varepsilon}^{b+\varepsilon} f(t)\,dt\right) = -\int_0^1 g(t)\phi_\varepsilon(t)\;dt.$$

Letting $\varepsilon \to 0$ and using the Lebesgue differentiation theorem for the left hand side and the Lebesgue dominated convergence theorem for the right hand side, we conclude that

$$f(a) - f(b) = -\int_a^b g(t)\;dt$$

for almost all $a, b \in (0,1),\ a < b$. Since $g \in L^1(0,1)$ the result follows.

Problem 3: Let $\{g_n\}$ be a sequence of measurable functions on $[0,1]$ such that

(a) $|g_n(x)| \le C$ for a.e. $x \in [0,1]$,

(b) $\lim_{n\to\infty} \int_0^a g_n(x)\;dx = 0$ for all $a \in (0,1)$.

Prove that if $f \in L^1(0,1)$ then

$$\lim_{n\to\infty} \int_0^1 f(x)g_n(x)\;dx = 0\ .$$

Solution: Let

$$V = \{f \in L^1(0,1) : \lim_{n\to\infty} \int_0^1 f(x)g_n(x)\;dx = 0\}.$$

Then V is a closed linear subspace of $L^1(0,1)$. Indeed, the linearity is obvious. To show that V is closed, take $f \in L^1(0,1)$, and let $h \in V$. Then

$$\left|\limsup_{n\to\infty} \int_0^1 f(x)g_n(x)\;dx\right| \le \limsup_{n\to\infty} \int_0^1 |f(x) - h(x)| \cdot |g_n(x)|\;dx \le C\|f-h\|_1.$$

If $f \in \text{cl}(V)$, then the right hand side can be made arbitrarily small which implies that $f \in V$.

By the assumption, $\mathbb{1}_{[1,a]} \in V$ for any $a \in [0,1]$. Hence, the indicator of any finite union of intervals in contained in E. Therefore, $\mathbb{1}_E \in F$ for any measurable set E as the indicators of such sets can be approximated arbitrarily well by the indicators of finite unions of intervals in L^1 norm. This in turn implies that any simple function belongs to V, and thus $V = L_1(0,1)$.

Problem 4: Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $\{f_n\}_{n=1}^{\infty} \subset L_2(\mu)$ be a sequence of functions such that $f_n \to f$ a.e. and$\|f_n\|_2 \le M$ for all $n \in \mathbb{N}$. Prove that $\int_X f_n \, d\mu \to \int_X f \, d\mu$.

Solution: Let $\varepsilon > 0$. By Egoroff's theorem, we can find $E \subset X$ with $\mu(E^c) < \varepsilon$ such that $f_n \to f$ uniformly on E. Then by Cauchy-Schwarz inequality,

$$\begin{aligned}\limsup_{n\to\infty} \|f_n - f\|_1 &\le \limsup_{n\to\infty} \int_E |f_n - f| \, d\mu + \sup_{n\to\infty} \int_{E^c} |f_n - f| \, d\mu \\ &\le \big(\mu(E^c)\big)^{1/2} \left(\int_{E^c} |f_n - f|^2 \, d\mu \right)^{1/2} \le 2M\sqrt{\varepsilon}.\end{aligned}$$

Since ε is arbitrary, the result follows.

Problem 5: Let $A \subset \{(x,y) \in \mathbb{R}^2 : \ |x| + |y| \le 1\}$ be a measurable set with the two-dimensional Lebesgue measure $m_2(A) \ge 1$. For $x \in [-1,1]$, denote $A_x = \{y \in [-1,1] : \ (x,y) \in A\}$. Prove that there exists $x \in [-1,1]$ such that

$$m_1(A_x) \ge 2 - \sqrt{2}.$$

Solution: Let $a \in [0,1]$. Then $m(A_x) \le 2(1 - |x|)$ for any $x \in [-1,1]$. Hence,

$$\begin{aligned}1 \le m_2(A) &= \int_{-1}^{-a} m(A_x) \, dx + \int_{-a}^{a} m(A_x) \, dx + \int_{a}^{1} m(A_x) \, dx \\ &\le 2 \int_a^1 2(1-x) \, dx + \int_{-a}^{a} m(A_x) \, dx \\ &\le 2(1-a)^2 + 2a \max_{x \in [-a,a]} m(A_x).\end{aligned}$$

This implies that

$$\max_{x \in [-a,a]} m(A_x) \ge \frac{1 - 2(1-a)^2}{2a}$$

for any $a \in (0,1)$. Optimizing the last expression over a, we get the required bound.