Department of Mathematics, University of Michigan
Analysis Qualifying Exam, January 8, 2022
Morning Session, 9.00 AM-12.00

Problem 1: Let E be a measurable subset of $[0,1]$. Suppose there exists $\alpha \in (0,1)$ such that

$$m(E \cap J) \ \geq \ \alpha \cdot m(J) \quad \text{for all subintervals } J \text{ of } [0,1] \ .$$

Prove that $m(E) = 1$.

Problem 2: Let $f, g \in L^1(0,1)$. Assume for all functions $\phi \in C^\infty([0,1])$ with $\phi(0) = \phi(1)$ that

$$\int_0^1 f(t)\phi'(t) \ dt \ = \ -\int_0^1 g(t)\phi(t) \ dt \ .$$

Show that $f(\cdot)$ is absolutely continuous and $f' = g$.

Problem 3: Let $\{g_n\}$ be a sequence of measurable functions on $[0,1]$ such that

(a) $|g_n(x)| \leq C$ for a.e. $x \in [0,1]$,

(b) $\lim_{n\to\infty} \int_0^a g_n(x) \ dx \ = \ 0$ for all $a \in (0,1)$.

Prove that if $f \in L^1(0,1)$ then

$$\lim_{n\to\infty} \int_0^1 f(x) g_n(x) \ dx \ = \ 0 \ .$$

Problem 4: Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $\{f_n\}_{n=1}^\infty \subset L_2(\mu)$ be a sequence of functions such that $f_n \to f$ a.e. and $\|f_n\|_2 \leq M$ for all $n \in \mathbb{N}$. Prove that $\int_X f_n \, d\mu \to \int_X f \, d\mu$.

Problem 5: Let $A \subset \{(x,y) \in \mathbb{R}^2 : \ |x| + |y| \leq 1\}$ be a measurable set with the two-dimensional Lebesgue measure $m_2(A) \geq 1$. For $x \in [-1,1]$, denote $A_x = \{y \in [-1,1] : \ (x,y) \in A\}$. Prove that there exists $x \in [-1,1]$ such that

$$m_1(A_x) \geq 2 - \sqrt{2}.$$