Department of Mathematics, University of Michigan
Analysis Qualifying Exam, January 9, 2022
Morning Session, 9.00 AM-12.00

Problem 1: Let $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ be the upper half plane in $\mathbb{C}$ and $f : \mathbb{H} \to \mathbb{C}$ an analytic function which satisfies $|f(z)| < 1$ for all $z \in \mathbb{H}$.

(a) Show that

$$|f'(i)| \ \le \frac{1 - |f(i)|^2}{2} \ .$$

(b) Identify all such analytic functions for which equality holds in (a).

Solution: (a) This is the same argument as gives Pick's lemma and so follows from the Schwarz lemma $|g'(0)| < 1$ for analytic functions $g : \mathbb{D} \to \mathbb{D}$ satisfying $g(0) = 0$. We map $\mathbb{D}$ to $\mathbb{H}$ using a fractional linear transformation f_1 which satisfies $f_1(0) = i$ and another one f_2 from $\mathbb{D} \to \mathbb{D}$ which maps $a = f(i)$ to 0. Then $g = f_2 \circ f \circ f_1$ and the inequality $|g'(0)| < 1$ becomes $|f_2'(a) f'(i) f_1'(0)| < 1$. We can take

$$f_1(z) = i\frac{1+z}{1-z} \ , \quad f_2(z) = \frac{z-a}{1-\bar{a}z} \ ,$$

and these have derivatives

$$f_1'(z) = \frac{2i}{(1-z)^2} \ , \quad f_2'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2} \ .$$

Hence $|f_1'(0)| = 2, \ |f_2'(a)| = [1-|a|^2]^{-1}$, which yields the inequality.

(b) Schwarz implies that if $|g'(0)| = 1$ then $g(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$, i.e. $g(\cdot)$ is a rotation. This gives a formula for $f(\cdot)$ by inverting the fractional linear transformations.

Problem 2: Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\ln|x^2-1|}{x^2+1}\, dx.$$

Solution: Consider a branch of logarithm with the cut along the ray $\{-it, \ t \ge 0\}$. Observe that

$$\int_{-\infty}^{\infty} \frac{\ln|x^2-1|}{x^2+1}\, dx \ = \ 2\int_{-\infty}^{\infty} \frac{\ln|x-1|}{x^2+1}\, dx \ = \ 2\Re \int_{-\infty}^{\infty} \frac{\ln(x-1)}{x^2+1}\, dx.$$

To calculate $I(\lambda)$ we use contour integration. Let $\varepsilon \in (0,1)$ and $R > 2$. Consider a contour Γ consisting of

$$\begin{aligned}
\Gamma_1 &= [1+\varepsilon, R], \\
\Gamma_2 &= \{Re^{it} :\ 0 \le t \le \pi\}, \\
\Gamma_3 &= [-R, 1-\varepsilon], \\
\Gamma_4 &= \{1+\varepsilon e^{it} :\ \pi \ge t \ge 0\}.
\end{aligned}$$

By the residue theorem,

$$\int_\Gamma \frac{\ln(z-1)}{z^2+1}\,dz = 2\pi i \mathrm{Res}\left(\frac{\ln(z-1)}{z^2+1}, i\right) = 2\pi i \frac{\ln(i-1)}{2i} = \pi\left(\ln(\sqrt{2}) + i\frac{3\pi}{4}\right).$$

Note that

$$\int_{\Gamma_1\cup\Gamma_3} \frac{\ln(z-1)}{z^2+1}\,dz \to \int_{-\infty}^{\infty} \frac{\ln(x-1)}{x^2+1}\,dx$$

as $\varepsilon \to 0$ and $R \to \infty$. Also,

$$\left|\int_{\Gamma_2} \frac{\ln(z-1)}{z^2+1}\,dz\right| \le \int_0^\pi \frac{\ln(R+1)+1}{R^2-1} \cdot R\,dt \to 0$$

as $R \to \infty$ and

$$\left|\int_{\Gamma_4} \frac{\ln(z-1)}{z^2+1}\,dz\right| \le \int_0^\pi \frac{\ln(1/\varepsilon)+t}{2-2\varepsilon} \cdot \varepsilon\,dt \to 0$$

as $\varepsilon \to 0$. Thus,

$$\int_{-\infty}^{\infty} \frac{\ln|x^2-1|}{x^2+1}\,dx \ = 2\Re\left(2\pi i \mathrm{Res}\left(\frac{\ln(z-1)}{z^2+1}, i\right)\right) = \pi \ln 2.$$

Problem 3: Let $f_0, \ldots, f_{n-1}$ be functions analytic in a neighborhood of a point $z_0 \in \mathbb{C}$, and let g be a function analytic in a punctured neighborhood of z_0. Define the function

$$h(z) = f_0(z) + f_1(z)g(z) + f_2(z)\big(g(z)\big)^2 + \cdots + f_{n-1}(z)\big(g(z)\big)^{n-1} + \big(g(z)\big)^n.$$

Show that if g has an essential singularity at z_0 then h has an essential singularity at z_0.

Solution: We assume for contradiction that $h(\cdot)$ does not have an essential singularity at z_0. Then there exists an integer $N \ge 0$ such that $\lim_{z\to z_0}(z-z_0)^N h(z) = 0$. We choose an integer $k \ge 0$ such that $nk \ge N$ and observe that

$$(z-z_0)^{nk}h(z) \ = \ \tilde{P}_n(\tilde{g}(z), z)\ , \quad \text{where } \tilde{g}(z) = (z-z_0)^k g(z)\ ,$$

and $\tilde{P}_n$ is the polynomial

$$\tilde{P}_n(w,z) \ = \ w^n + (z-z_0)^k f_{n-1}(z) w^{n-1} + \cdots + (z-z_0)^{(n-1)k} f_1(z) w + (z-z_0)^{nk} f_0(z)\ .$$

Observe that $\tilde{g}(\cdot)$ also has an essential singularity at z_0 and let $w \in \mathbb{C}$ be such that $\tilde{P}_n(w, z_0) \neq 0$. By the Casorati-Weierstrass theorem there exists a sequence z_m such that $\lim_{m\to\infty} z_m = z_0$ and $\lim_{m\to\infty} \tilde{g}(z_m) = w$. This implies $\lim_{n\to\infty}(z_m - z_0)^{nk} h(z_m) \neq 0$, a contradiction.

Problem 4: Let $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1,\ \Re z + \Im z > 1\}$. Find a conformal mapping $f(\cdot)$ from $\mathcal{D}$ onto the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. You may express $f(\cdot)$ as a composition of simpler maps.

Solution: $\partial\mathcal{D}$ is the intersection of a circle and a line with intersection points $1, i$. We use a fractional linear transformation to map 1 to 0 and i to ∞. One such map is

$$f_1(z) = \frac{z-1}{z-i} .$$

Since we have $f_1(-i) = (1-i)/2$ and $f_1(1/2 + i/2) = -1$, we see that $f_1(\mathcal{D}) = \mathcal{D}_1$ is the wedge of acute angle $\pi/4$ between the negative real axis and the half line from the origin though the point $-1+i$. We rotate and reflect the wedge to be symmetric about the positive real axis so we define f_2 by $f_2(z) = -e^{\pi i/8} z$. Next we set $f_3(z) = z^4$ which maps the wedge of angle $\pi/4$ symmetric about the positive real axis to the right half plane $\{\Re z > 0\}$. Finally we map $\{\Re z > 0\}$ to $\{|z| < 1\}$ by a fractional linear transformation such as

$$f_4(z) = \frac{z-1}{z+1} .$$

The final transformation is then $f = f_4 \circ f_3 \circ f_2 \circ f_1$.

Problem 5: Show that if $\Re\lambda > 1$ then the equation $e^z = z + \lambda$ has exactly one solution in the left half plane $\Re z < 0$.

Solution: Let $\mathcal{D}_\lambda$ be the disk $\{z \in \mathcal{C} :\ |z + \lambda| < 1\}$. Since $\Re\lambda > 1$ the closure of $\mathcal{D}_\lambda$ is in the left hand plane. Further if $z \notin \mathcal{D}_\lambda$ and $\Re z < 0$ then $f(z) = z + \lambda - e^z$ satisfies $|f(z)| \geq |z + \lambda| - |e^z| > 0$. Hence the only zeros of $f(\cdot)$ in the left hand plane lie in $\mathcal{D}_\lambda$. We let $g(z) = z + \lambda$ so $g(\cdot)$ has exactly one solution in $\mathcal{D}_\lambda$. Furthermore $|f(z) - g(z)| = |e^z| < |g(z)|$ for $z \in \partial\mathcal{D}_\lambda$. The Rouché theorem implies then that $f(\cdot)$ has the same number of zeros in $\mathcal{D}_\lambda$ as $g(\cdot)$.