Department of Mathematics, University of Michigan
Analysis Qualifying Exam, January 9, 2022
Morning Session, 9.00 AM-12.00

Problem 1: Let $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ be the upper half plane in $\mathbb{C}$ and $f : \mathbb{H} \to \mathbb{C}$ an analytic function which satisfies $|f(z)| < 1$ for all $z \in \mathbb{H}$.

(a) Show that

$$|f'(i)| \ \le \frac{1 - |f(i)|^2}{2} \ .$$

(b) Identify all such analytic functions for which equality holds in (a).

Problem 2: Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\ln|x^2 - 1|}{x^2 + 1}\, dx.$$

Problem 3: Let $f_0, \ldots, f_{n-1}$ be functions analytic in a neighborhood of a point $z_0 \in \mathbb{C}$, and let g be a function analytic in a punctured neighborhood of z_0. Define the function

$$h(z) = f_0(z) + f_1(z)g(z) + f_2(z)\big(g(z)\big)^2 + \cdots + f_{n-1}(z)\big(g(z)\big)^{n-1} + \big(g(z)\big)^n.$$

Show that if g has an essential singularity at z_0 then h has an essential singularity at z_0.

Problem 4: Let $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1,\ \Re z + \Im z > 1\}$. Find a conformal mapping $f(\cdot)$ from $\mathcal{D}$ onto the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. You may express $f(\cdot)$ as a composition of simpler maps.

Problem 5: Show that if $\Re\lambda > 1$ then the equation $e^z = z + \lambda$ has exactly one solution in the left half plane $\Re z < 0$.