Problem 1: Let \( \mu \) be a finite Borel measure on \([0, 1]\) and \( f : [0, 1] \to [0, \infty) \) an integrable function with respect to \( \mu \). Suppose further that
\[
\int_A |f| \, d\mu \leq \sqrt{\mu(A)} \quad \text{for all Borel sets } A \subset [0, 1].
\]
Prove that \( |f|^p \) is integrable with respect to \( \mu \) provided \( 1 \leq p < 2 \).

Solution: For \( n = 1, 2, \ldots \), let \( A_n = \{ x \in (0, 1) : 2^n < f(x) \leq 2^{n+1} \} \).

\[
2^n \mu(A_n) \leq \int_{A_n} |f| \, d\mu \leq \sqrt{\mu(A_n)}
\]
We conclude that
\[
\int_{[0, 1]} |f|^p \, d\mu \leq \mu([0, 1]) + \sum_{n=1}^{\infty} 2^{(n+1)p-2n} < \infty
\]
if \( p < 2 \).

Problem 2: Let \( f : (0, 1) \to \mathbb{R} \) be a Lebesgue measurable function which satisfies the inequality \( \int_0^1 t^3f(t)^4 \, dt < \infty \). Prove that
\[
\lim_{x \to 0} \frac{1}{|\log x|^{3/4}} \int_x^1 f(t) \, dt = 0.
\]

Solution: Using the Hölder inequality we have that
\[
\int_x^1 |f(t)| \, dt \leq \left[ \int_x^1 \frac{dt}{t} \right]^{3/4} \left[ \int_0^1 t^3 f(t)^4 \, dt \right]^{1/4} \leq C |\log x|^{3/4}.
\]
Generalizing this we have that
\[
\int_x^\delta |f(t)| \, dt \leq C_\delta |\log x|^{3/4}, \quad 0 < x < \delta,
\]
where \( C_\delta = \left[ \int_0^\delta t^3 f(t)^4 \, dt \right]^{1/4} \).

By the dominated convergence theorem we have \( \lim_{\delta \to 0} C_\delta = 0 \). The result follows by observing that
\[
\limsup_{x \to 0} \frac{1}{|\log x|^{3/4}} \int_x^1 |f(t)| \, dt \leq \limsup_{x \to 0} \frac{1}{|\log x|^{3/4}} \int_\delta^1 |f(t)| \, dt + C_\delta = C_\delta.
\]

Problem 3: Suppose \( A \) is a Lebesgue measurable subset of \( \mathbb{R} \) with positive measure \( m(A) > 0 \). Show that for any \( b \) with \( 0 < b < m(A) \) there exists a compact subset \( K \subset A \) with \( m(K) = b \).
Problem 4: Suppose \( - \) the requisite compact set is then \( x \) \( - \) theorem \( y \) that for all \( \) now define \( g = \sup \{ m(F) \} \), where the supremum is taken over all closed subsets \( F \) of \( A \). Since \( A \) is bounded the sets \( F \) are compact. Hence there exists compact \( K \subset A \) such that \( b < m(K) \). Now define \( g : (0, \infty) \to \mathbb{R}^+ \) by \( g(x) = m([-x, x] \cap K) \). By the monotone convergence theorem \( g(\cdot) \) is continuous and \( \lim_{x \to 0} g(x) = 0, \lim_{x \to \infty} g(x) = m(K) > b \). The intermediate value theorem implies there exists \( x_b > 0 \) such that \( g(x_b) = b \). The requisite compact set is then \([-x_b, x_b] \cap K\).

**Solution:** First we reduce to the case when \( A \) is bounded. Since \( \lim_{N \to \infty} m([-N, N] \cap A) = m(A) > b \), it follows that there exists \( N \geq 1 \) such that \( m([-N, N] \cap A) > b \). Hence we may replace the possibly unbounded \( A \) in the problem with the bounded set \([-N, N] \cap A \). Next by inner regularity of \( A \) one has \( m(A) = \sup_{F \subset A} m(F) \), where the supremum is taken over all closed subsets \( F \) of \( A \). Since \( A \) is bounded the sets \( F \) are compact. Hence there exists compact \( K \subset A \) such that \( b < m(K) \). Now define \( g : (0, \infty) \to \mathbb{R}^+ \) by \( g(x) = m([-x, x] \cap K) \). By the monotone convergence theorem \( g(\cdot) \) is continuous and \( \lim_{x \to 0} g(x) = 0, \lim_{x \to \infty} g(x) = m(K) > b \). The intermediate value theorem implies there exists \( x_b > 0 \) such that \( g(x_b) = b \). The requisite compact set is then \([-x_b, x_b] \cap K\).

Problem 4: Suppose \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function and \( k \) an integer such that for all \( y \in \mathbb{R} \) the number of distinct solutions to the equation \( f(x) = y \) is bounded by \( k \). Prove that the derivative \( f'(x) \) exists for a.e. \( x \in \mathbb{R} \).

**Solution:** Let \([a, b] \subset \mathbb{R} \) be a compact interval such that \( m = \inf_{[a, b]} f(\cdot) \) and \( M = \sup_{[a, b]} f(\cdot) \). Let \( a_1 = \inf\{ x \in [a, b] : f(x) = m \} \) and \( b_1 = \inf\{ x \in [a, b] : f(x) = M \} \}. We may assume wlog that \( a_1 < b_1 \). Now define \( g_1 : [m, M] \to [a_1, b_1] \) by \( g_1(y) = \inf\{ x \in [a_1, b_1] : f(x) = y \} \). The function \( g_1 \) is strictly monotonic increasing and \( g_1([m, M]) = [a_1, b_1] \subset [a, b] \). Hence \( f \) is strictly monotonic increasing on \([a_1, b_1] \). It follows that \( f'(\cdot) \) is differentiable a.e. on \([a_1, b_1] \). We can proceed similarly with \( f \) on the intervals \([a_1, b_1] \) and \([b_1, b] \), until after a finite number of steps we conclude that \( f(\cdot) \) is differentiable a.e. on \([a, b] \).

Alternatively we can show by contradiction that \( f(\cdot) \) is BV on \([a, b] \). Let \( m = \inf_{[a, b]} f(\cdot) \) and \( M = \sup_{[a, b]} f(\cdot) \). Since \( f(\cdot) \) is not BV on \([a, b] \) there exist \( a \leq x_1 < x_2 < \cdots < x_N \leq b \) such that
\[
\sum_{j=1}^{N-1} |f(x_{j+1}) - f(x_j)| \geq k(M - m) + 1.
\]
Let \( S_j \) be the set \( f((x_j, x_{j+1})) \), \( j = 1, \ldots, N-1 \). Since the open sets \((x_j, x_{j+1}) \), \( j = 1, \ldots, N-1 \) are disjoint the assumption of the problem implies that
\[
\sum_{j=1}^{N-1} \chi_{S_j} \leq k,
\]
where \( \chi_S \) denotes characteristic function of \( S \). Since \( S_j \subset [m, M] \), \( j = 1, \ldots, N-1 \) it then follows that
\[
\sum_{j=1}^{N-1} |f(x_{j+1}) - f(x_j)| \leq \sum_{j=1}^{N-1} m(S_j) \leq k(M - m),
\]
which contradicts our initial inequality.
**Problem 5:** Let $f$ be in $L^1(\mathbb{R})$ and denote by $Mf$ the restricted maximal function

$$Mf(x) = \max_{0 < t < 1} \frac{1}{2t} \int_{x-t}^{x+t} |f(x')| \, dx' , \quad x \in \mathbb{R}.$$ 

Prove that

$$M(f \ast g)(x) \leq Mf \ast Mg(x) , \quad x \in \mathbb{R}, \ f, g \in L^1(\mathbb{R}),$$

where the operation $\ast$ denotes convolution:

$$f \ast g(x) = \int_{\mathbb{R}} f(x - y)g(y) \, dy , \quad x \in \mathbb{R}.$$

**Solution:** We may assume wlog that $f, g$ are non-negative. Then from the Lebesgue differentiation theorem we have that $Mg(x) \geq g(x)$ for a.e. $x$. Also

$$\int_{x-t}^{x+t} f(x') \, dx' = \chi_t \ast f(x) , \quad \text{where} \ \chi_t(y) = 1 \text{ if } |y| < t, \ \chi_t(y) = 0 \text{ if } |y| > t.$$

Now we use the associative property of convolutions. Thus

$$\chi_t \ast [f \ast g](x) = [\chi_t \ast f] \ast g(x).$$

This yields the inequality $M(f \ast g)(x) \leq Mf(x) \ast g(x)$. We may avoid use of the Lebesgue theorem by observing that

$$\chi_{t-s} \leq \chi_t \ast \frac{1}{2s} \chi_s \quad \text{for all } 0 < s < t.$$

Since the operation of convolution is also commutative we have that

$$\chi_{t-s} \ast [f \ast g](x) \leq [\chi_t \ast f] \ast \left[ \frac{1}{2s} \chi_s \ast g \right](x).$$