Department of Mathematics, University of Michigan
Real Analysis Qualifying Exam
May 4, 2023, 9.00 am-12.00 pm

Problem 1: Let μ be a finite Borel measure on $[0,1]$ and $f : [0,1] \to [0,\infty)$ an integrable function with respect to μ. Suppose further that

$$\int_A |f| d\mu \ \le \ \sqrt{\mu(A)} \quad \text{for all Borel sets } A \subset [0,1] \ .$$

Prove that $|f|^p$ is integrable with respect to μ provided $1 \le p < 2$.
Solution: For $n = 1, 2, \dots$, let $A_n = \{x \in (0,1) : 2^n < f(x) \le 2^{n+1}\}$.

$$2^n \mu(A_n) \ \le \ \int_{A_n} |f| \ d\mu \ \le \ \sqrt{\mu(A_n)}$$

We conclude that

$$\int_{[0,1]} |f|^p d\mu \ \le \ \mu([0,1]) + \sum_{n=1}^{\infty} 2^{(n+1)p} \mu(A_n) \ \le \ \mu([0,1]) + \sum_{n=1}^{\infty} 2^{(n+1)p-2n} \ < \ \infty$$

if $p < 2$.

Problem 2: Let $f : (0,1) \to \mathbb{R}$ be a Lebesgue measurable function which satisfies the inequality $\int_0^1 t^3 f(t)^4 \ dt < \infty$. Prove that

$$\lim_{x \to 0} \frac{1}{|\log x|^{3/4}} \int_x^1 f(t) \ dt \ = \ 0 \ .$$

Solution: Using the Hölder inequality we have that

$$\int_x^1 |f(t)| \ dt \ \le \ \left[\int_x^1 \frac{dt}{t} \right]^{3/4} \left[\int_0^1 t^3 f(t)^4 \ dt \right]^{1/4} \ \le \ C |\log x|^{3/4} \ .$$

Generalizing this we have that

$$\int_x^\delta |f(t)| \ dt \ \le \ C_\delta |\log x|^{3/4} \ , \ 0 < x < \delta, \quad \text{where } C_\delta = \left[\int_0^\delta t^3 f(t)^4 \ dt \right]^{1/4} \ .$$

By the dominated convergence theorem we have $\lim_{\delta \to 0} C_\delta = 0$. The result follows by observing that

$$\limsup_{x \to 0} \frac{1}{|\log x|^{3/4}} \int_x^1 |f(t)| \ dt \le \limsup_{x \to 0} \frac{1}{|\log x|^{3/4}} \int_\delta^1 |f(t)| \ dt + C_\delta = C_\delta \ .$$

Problem 3: Suppose A is a Lebesgue measurable subset of $\mathbb{R}$ with positive measure $m(A) > 0$. Show that for any b with $0 < b < m(A)$ there exists a compact subset $K \subset A$ with $m(K) = b$.

Solution: First we reduce to the case when A is bounded. Since $\lim_{N\to\infty} m([-N,N]\cap A) = m(A) > b$, it follows that there exists $N \ge 1$ such that $m([-N,N]\cap A) > b$. Hence we may replace the possibly unbounded A in the problem with the bounded set $[-N,N]\cap A$. Next by inner regularity of A one has $m(A) = \sup_{F\subset A} m(F)$, where the supremum is taken over all closed subsets F of A. Since A is bounded the sets F are compact. Hence there exists compact $K \subset A$ such that $b < m(K)$. Now define $g : (0,\infty) \to \mathbb{R}^+$ by $g(x) = m([-x,x]\cap K)$. By the monotone convergence theorem $g(\cdot)$ is continuous and $\lim_{x\to 0} g(x) = 0, \lim_{x\to\infty} g(x) = m(K) > b$. The intermediate value theorem implies there exists $x_b > 0$ such that $g(x_b) = b$. The requisite compact set is then $[-x_b, x_b]\cap K$.

Problem 4: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and k an integer such that for all $y \in \mathbb{R}$ the number of distinct solutions to the equation $f(x) = y$ is bounded by k. Prove that the derivative $f'(x)$ exists for a.e. $x \in \mathbb{R}$.

Solution: Let $[a,b] \subset \mathbb{R}$ be a compact interval such that $m = \inf_{[a,b]} f(\cdot)$ and $M = \sup_{[a,b]} f(\cdot)$. Let $a_1 = \inf\{x \in [a,b] : f(x) = m\}$ and $b_1 = \inf\{x \in [a,b] : f(x) = M\}$. We may assume wlog that $a_1 < b_1$. Now define $g_1 : [m,M] \to [a_1,b_1]$ by $g_1(y) = \inf\{x \in [a_1,b_1] : f(x) = y\}$. The function g_1 is strictly monotonic increasing and $g_1([m,M]) = [a_1,b_1] \subset [a,b]$. Hence f is strictly monotonic increasing on $[a_1,b_1]$. It follows that $f'(\cdot)$ is differentiable a.e. on $[a_1,b_1]$. We can proceed similarly with f on the intervals $[a,a_1]$ and $[b_1,b]$, until after a finite number of steps we conclude that $f(\cdot)$ is differentiable a.e. on $[a,b]$.

Alternatively we can show by contradiction that $f(\cdot)$ is BV on $[a,b]$. Let $m = \inf_{[a,b]} f(\cdot)$ and $M = \sup_{[a,b]} f(\cdot)$. Since $f(\cdot)$ is not BV on $[a,b]$ there exist $a \le x_1 < x_2 < \cdots < x_N \le b$ such that

$$\sum_{j=1}^{N-1} |f(x_{j+1}) - f(x_j)| \ \ge k(M-m) + 1 \ .$$

Let S_j be the set $f((x_j, x_{j+1})),\ j = 1,\dots,N-1$. Since the open sets $(x_j, x_{j+1}),\ j = 1,\dots,N-1$ are disjoint the assumption of the problem implies that

$$\sum_{j=1}^{N-1} \chi_{S_j} \le k \ ,$$

where χ_S denotes characteristic function of S. Since $S_j \subset [m,M],\ j = 1,\dots,N-1$ it then follows that

$$\sum_{j=1}^{N-1} |f(x_{j+1}) - f(x_j)| \ \le \ \sum_{j=1}^{N-1} m(S_j) \ \le \ k(M-m) \ ,$$

which contradicts our initial inequality.

Problem 5: Let f be in $L^1(\mathbb{R})$ and denote by Mf the restricted maximal function

$$Mf(x) \ = \ \max_{0<t<1} \frac{1}{2t} \int_{x-t}^{x+t} |f(x')| \ dx' \ , \quad x \in \mathbb{R} \ .$$

Prove that

$$M(f*g)(x) \ \le Mf * Mg(x) \ , \quad x \in \mathbb{R}, \ f,g \in L^1(\mathbb{R}),$$

where the operation $*$ denotes convolution:

$$f*g(x) \ = \int_{\mathbb{R}} f(x-y)g(y) \ dy \ , \quad x \in \mathbb{R}.$$

Solution: We may assume wlog that f,g are non-negative. Then from the Lebesgue differentiation theorem we have that $Mg(x) \ge g(x)$ for a.e. x. Also

$$\int_{x-t}^{x+t} f(x') \ dx' \ = \ \chi_t * f(x) \ , \quad \text{where } \chi_t(y) = 1 \text{ if } |y| < t, \quad \chi_t(y) = 0 \text{ if } |y| > t.$$

Now we use the associative property of convolutions. Thus

$$\chi_t * [f*g](x) \ = \ [\chi_t * f] * g(x) \ .$$

This yields the inequality $M(f*g)(x) \le Mf(x) * g(x)$. We may avoid use of the Lebesgue theorem by observing that

$$\chi_{t-s} \le \chi_t * \frac{1}{2s}\chi_s \quad \text{for all } 0 < s < t \ .$$

Since the operation of convolution is also commutative we have that

$$\chi_{t-s} * [f*g](x) \ \le \ [\chi_t * f] * [\frac{1}{2s}\chi_s * g](x) \ .$$