Department of Mathematics, University of Michigan
Complex Analysis Qualifying Exam
May 3, 2023, 2.00 pm-5.00 pm

Problem 1: (a) Let $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ be the unit disk and $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function satisfying $\Re f(z) > 0, \ z \in \mathbb{D}$. Show that $|f'(0)| \le 2\Re f(0)$.
(b) Suppose instead that $f(\mathbb{D}) \subset \mathbb{D} - \{0\}$. Prove that $|f'(0)| \le 2e^{-1}$.

Solution: (a) This follows from the Schwarz lemma by mapping the right half plane to $\mathbb{D}$ such that $f(0) \to 0$. The mapping is $w \to [w - f(0)][w + \overline{f(0}]$. By Schwarz the function $g(\cdot)$ satisfies $|g'(0)| \le 1$. We have

$$g(z) = \frac{f(z) - f(0)}{f(z) + \overline{f(0)}} = 1 - \frac{[f(0) + \overline{f(0)}]}{f(z) + \overline{f(0)}}, \quad g'(z) = \frac{[f(0) + \overline{f(0)}]f'(z)}{[f(z) + \overline{f(0)}]^2} \ .$$

We conclude that

$$|f'(0)| \ \le \ f(0) + \overline{f(0)} \ = \ 2\Re f(0) \ .$$

(b) Note that $h(z) = -\log f(z)$ is well defined from Cauchy's theorem by the integral formula

$$h(z) = -\int_0^z \frac{f'(z')}{f(z')} \ dz' + \text{constant} \ ,$$

for a suitable constant such that $\Re h(z) = -\log |f(z)|$. . Then $h(\cdot)$ is holomorphic on $\mathbb{D}$ and $\Re h(\cdot) > 0$. Hence by (a) $|h'(0)| \le 2\Re h(0)$, which implies that $|f'(0)| \le 2\max_{0<\alpha<1} \alpha |\log \alpha| = 2e^{-1}$.

Problem 2: Use contour integration to evaluate the integral

$$\int_0^\infty \frac{\log x}{(x+1)^2\sqrt{x}} \ dx \ .$$

Solution: We make the substitution $x = z^2$, whence it is sufficient to evaluate the integral

$$4\int_0^\infty \frac{\log z}{(z^2+1)^2} \ dz \ = \ 2\Re \int_{-\infty}^\infty \frac{\log z}{(z^2+1)^2} \ dz \ .$$

Now we use the residue theorem for the region bounded by the contour Γ_R which consists of the line segment $\{z \in \mathbb{R} : \ -R \le z \le R\}$ and the semicircle $\{z \in \mathbb{C} : \Im z > 0, \ |z| = R\}$. Then if $R > 1$ one has

$$\int_{\Gamma_R} \frac{\log z}{(z^2+1)^2} \ dz \ = \ 2\pi i \ \text{Res}\left(\frac{\log z}{(z^2+1)^2}, \ i\right) \ .$$

To evaluate the residue we write

$$\frac{\log z}{(z^2+1)^2} \ = \ \frac{f(z)}{(z-i)^2} \ = \ \frac{f(i)}{(z-i)^2} + \frac{f'(i)}{(z-i)} + \cdots \ , \quad f(z) = \frac{\log z}{(z+i)^2} \ .$$

We have that

$$f'(z) = \frac{1}{z(z+i)^2} - \frac{2\log z}{(z+i)^3}\ , \quad f'(i) = \frac{i}{4} + \frac{\pi}{8}\ .$$

Hence the integral over Γ_R is $-\pi/2 + \pi^2 i/4$. Finally we note that

$$\left| \int_{\Im z>0, |z|=R} \frac{\log z}{(z^2+1)^2}\ dz \right| \ \le\ \frac{\pi R \log R}{(R^2-1)^2} \to 0 \ \text{ as } R \to \infty\ .$$

Hence the value of the integral in the problem is $-\pi$.

Problem 3: Find a conformal mapping from the unit disk $\mathbb{D} = \{z \in \mathbb{C} :\ |z| < 1\}$ onto the region $\mathbb{U} = \{z = x + iy \in \mathbb{C} :\ \ y < x^2\}$. You may write your solution as a composition of simpler maps. Make sure to explain why each of your simpler maps is conformal.

Solution: We need to map the unit circle onto the parabola $y = x^2$. We first begin by finding a map which takes a line onto the parabola. Consider the line $\Re z = \alpha$, so $\{\alpha + iy \in \mathbb{C} :\ \ y \in \mathbb{R}\}$. The image under the mapping $w = z^2$ is $\{\alpha^2 - y^2 + 2i\alpha y \in \mathbb{C} :\ \ y \in \mathbb{R}\}$. Writing $w = \xi + i\eta$ the image is $\{\xi + i\eta \in \mathbb{C} :\ \eta^2 = 4\alpha^2(\alpha^2 - \xi)\}$. Taking $\alpha = 1/2$ this becomes $\eta^2 = 1/4 - \xi$. We can map this to the parabola $y = x^2$ by translation and rotation, so $w_1 = w - 1/4$ and $w_2 = -iw_1$. Noting that 0 maps to $i/4$ we see that the region $\Re z > 1/2$ maps to the region $y < x^2$. Finally we need to map $\mathbb{D}$ onto the half plane $\Re z > 1/2$. Note that $w = (1-z)/(1+z)$ takes $\mathbb{D}$ onto the half plane $\Re w > 0$, whence the mapping $w = (1-z)/(1+z) + 1/2$ maps to the region $\Re w > 1/2$. The only one of these mappings which is not necessarily conformal is the 2-1 mapping $w = z^2$. However it does map the region $\Re z > 1/2$ conformally.

Problem 4: Let $f(\cdot)$ be a meromorphic function on $\mathbb{C}$ with a finite number of zeros and poles. Assume further there are constants A, C with $A \ne 0$ such that

$$|f(z) - A| \ \le\ \frac{C}{|z|^2} \quad \text{for all large } |z|\ .$$

(a) Prove that $f(\cdot)$ is a rational function.
(b) Suppose the poles and zeros of $f(\cdot)$ in $\mathbb{C}$ are $z_1, \dots, z_k$, with corresponding multiplicities $m_1, \dots, m_k \in \mathbb{Z}$. Show that $m_1 z_1 + \cdots m_k z_k = 0$.

Solution: (a) Since $\lim_{|z|\to\infty} f(z) = A$ the function $f(\cdot)$ is meromorphic on the Riemann sphere and therefore a rational function.
(b) We have from the argument principle that

$$\frac{1}{2\pi i} \int_{|z|=R} \frac{z f'(z)}{f(z)}\ dz \ =\ \sum_{j=1}^{k} m_j z_j \quad \text{if } R > \max\{|z_j| :\ 1 \le j \le k\}\ .$$

The Laurent expansion of $f(\cdot)$ about $z = \infty$ is $f(z) = A + a_2/z^2 + a_3/z^3 + \dots$. Hence

$$\left|\frac{zf'(z)}{f(z)}\right| \le \frac{4|a_2|}{A|z|^2} \quad \text{if } |z| = R, \quad \text{for all sufficiently large } R.$$

We conclude that

$$\left|\frac{1}{2\pi i}\int_{|z|=R} \frac{zf'(z)}{f(z)}\, dz\right| \le \frac{4|a_2|}{AR} \to 0 \qquad R \to \infty \ .$$

Problem 5: Let $\mathcal{D} \subset \mathbb{C}$ be a domain (open and connected), and $f_n : \mathcal{D} \to \mathbb{C},\ n = 1, 2, \dots,$ a sequence of holomorphic functions. Suppose further there is a continuous function $f : \mathcal{D} \to \mathbb{C}$ such that

$$\lim_{n\to\infty} \int_D |f_n(x+iy) - f(x+iy)|\, dx\, dy \ = \ 0 \quad \text{on all disks } \mathrm{D} \subset \mathcal{D} \ .$$

Prove that f is a holomorphic function.

Solution: We use the Cauchy theorem. Thus suppose $\mathcal{D}$ contains the disk $\{z \in \mathbb{C} : |z - z_0| < r_2\}$ and $0 < r_1 < r_2$. Then if $|z| < r_1 < r_2$, one has

$$\begin{aligned} f_n(z + z_0) \ &= \ \frac{1}{2\pi(r_2 - r_1)} \int_{r_1}^{r_2} \int_0^{2\pi} \frac{f_n(z_0 + re^{i\theta})\ re^{i\theta} dr d\theta}{z - re^{i\theta}} \\ &= \ \frac{1}{2\pi(r_2 - r_1)} \int_{r_1^2 < x^2+y^2 < r_2^2} \frac{f_n(z_0 + x + iy)\ (x+iy) dx dy}{\sqrt{x^2+y^2}[z - (x+iy)]} \ . \end{aligned}$$

From the assumption of the problem and the above representation the functions $z \to f_n(z + z_0), |z| < r_1 - \varepsilon,\ n = 1, 2, \dots,$ form a uniformly Cauchy sequence for any $\varepsilon > 0$. Hence by the Weierstrasse theorem there exists a holomorphic function $g(\cdot)$ on the disk $\{|z| < r_1\}$ such that $\lim_{n\to\infty} f_n(z + z_0) = g(z)$. To show $f(z + z_0) = g(z)$ we observe from our limiting procedure that

$$\int_D |g(x+iy) - f(z_0 + x + iy)|\, dx\, dy \ = \ 0 \quad \text{on the disk } D = \{|z| < r_1\} \ .$$