Department of Mathematics, University of Michigan
Complex Analysis Qualifying Exam
May 3, 2023, 2.00 pm-5.00 pm

Problem 1: (a) Let $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ be the unit disk and $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function satisfying $\Re f(z) > 0, \ z \in \mathbb{D}$. Show that $|f'(0)| \le 2\Re f(0)$.
(b) Suppose instead that $f(\mathbb{D}) \subset \mathbb{D} - \{0\}$. Prove that $|f'(0)| \le 2e^{-1}$.

Problem 2: Use contour integration to evaluate the integral

$$\int_0^\infty \frac{\log x}{(x+1)^2\sqrt{x}}\, dx \ .$$

Problem 3: Find a conformal mapping from the unit disk $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ onto the region $\mathbb{U} = \{z = x + iy \in \mathbb{C} : \ y < x^2\}$. You may write your solution as a composition of simpler maps. Make sure to explain why each of your simpler maps is conformal.

Problem 4: Let $f(\cdot)$ be a meromorphic function on $\mathbb{C}$ with a finite number of zeros and poles. Assume further there are constants A, C with $A \neq 0$ such that

$$|f(z) - A| \ \le \ \frac{C}{|z|^2} \quad \text{for all large } |z| \ .$$

(a) Prove that $f(\cdot)$ is a rational function.
(b) Suppose the poles and zeros of $f(\cdot)$ in $\mathbb{C}$ are $z_1, \ldots, z_k$, with corresponding multiplicities $m_1, \ldots, m_k \in \mathbb{Z}$. Show that $m_1 z_1 + \cdots m_k z_k = 0$.

Problem 5: Let $\mathcal{D} \subset \mathbb{C}$ be a domain (open and connected), and $f_n : \mathcal{D} \to \mathbb{C}, \ n = 1, 2, \ldots$, a sequence of holomorphic functions. Suppose further there is a continuous function $f : \mathcal{D} \to \mathbb{C}$ such that

$$\lim_{n\to\infty} \int_D |f_n(x+iy) - f(x+iy)|\, dx\, dy \ = \ 0 \quad \text{on all disks } \mathrm{D} \subset \mathcal{D} \ .$$

Prove that f is a holomorphic function.