Department of Mathematics, University of Michigan
Complex Analysis Qualifying Exam
January 8, 2023, 2.00 pm-5.00 pm

Problem 1: Use contour integration to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{x^2 - x} \ .$$

Problem 2: Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function from the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to the right half plane $\{w \in \mathbb{C} : \Re w > 0\}$ with the property $f(0) = 1$. Prove that

$$|f(z)| \ \le \ \frac{1 + |z|}{1 - |z|} \quad \text{for } z \in \mathbb{D} \ .$$

Problem 3: Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which has the property $f(-z) = -f(z), \ z \in \mathbb{D}$.
a) Show there is a holomorphic function $g : \mathbb{D} \to \mathbb{C}$ such that $g(z^2) = [f(z)]^2, \ z \in \mathbb{D}$.
b) Prove that if $f(\cdot)$ is one to one then the function g in a) is one to one.

Problem 4: An entire function $f : \mathbb{C} \to \mathbb{C}$ is of exponential type if there exist constants $C_1, C_2 > 0$ such that $|f(z)| \le C_1 e^{C_2|z|}, \ z \in \mathbb{C}$. Show that a function $f(\cdot)$ is of exponential type if and only if its derivative $f'(\cdot)$ is of exponential type.

Problem 5: Let $\mathcal{D} \subset \mathbb{C}$ be a domain i.e. open and connected, and $f, g : \mathcal{D} \to \mathbb{C}$ holomorphic functions which have the property that $|f(z)| + |g(z|$ is constant for $z \in \mathcal{D}$. Prove that the functions f and g are constant.
Hint: Apply the maximum principal judiciously.