Department of Mathematics, University of Michigan
Analysis Qualifying Exam, May 4, 2022
Morning Session, 9.00 AM-12.00

Problem 1: Suppose $f : (0,1) \to \mathbb{R}$ is integrable and define a function $g : (0,1) \to \mathbb{R}$ by

$$g(x) \;=\; \int_x^1 \frac{f(t)}{t}\, dt \;, \quad 0 < x < 1.$$

Prove that g is also integrable.

Problem 2: Let $f : [0,1] \to \mathbb{R}$ be a positive function such that f and $1/f$ are integrable. Prove that $\log f$ is integrable and

$$\lim_{q\to\infty} q \cdot \left(\int_0^1 f(x)^{1/q}\, dx - 1 \right) \;=\; \int_0^1 \log f(x)\, dx \;.$$

Problem 3: Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Let $\mathcal{C} \subset \mathcal{A}$ be a sub-sigma algebra of $\mathcal{A}$. Prove that for any $f \in L^1(\mu)$ there exists a $\mathcal{C}-$ measurable integrable function g such that

$$\int_E g\, d\mu \;=\; \int_E f\, d\mu \quad \text{for any } E \in \mathcal{C} \;.$$

Problem 4: Let $f_n : [0,1] \to \mathbb{R},\ n = 1, 2, \ldots,$ be a sequence of non-negative

Lebesgue measurable functions such that $\lim_{n\to\infty} f_n(x) = 0$ for almost every $x \in [0,1]$. Prove there exists an infinite subsequence $f_{n_k},\ k = 1, 2, \ldots,$ such that the series

$$\sum_{k=1}^{\infty} f_{n_k}(x) \quad \text{converges for almost every } x \in [0,1] \;.$$

Hint: Use Egorov's theorem.

Problem 5: Suppose for $n = 1, 2, \ldots,$ the functions $F_n : [a,b] \to \mathbb{R}$ are increasing and nonnegative, and that the function F with domain $[a,b]$ defined by

$$F(x) \;=\; \sum_{n=1}^{\infty} F_n(x) \;,$$

is finite for all $x \in [a,b]$. Prove that the derivative $F'(x)$ exists a.e. and

$$F'(x) \;=\; \sum_{n=1}^{\infty} F_n'(x) \text{ for almost every } x \in [a,b].$$