Department of Mathematics, University of Michigan
Analysis Qualifying Exam, May 4, 2022
Morning Session, 9.00 AM-12.00

Problem 1: Suppose $f : (0,1) \to \mathbb{R}$ is integrable and define a function $g : (0,1) \to \mathbb{R}$ by

$$g(x) \ = \ \int_x^1 \frac{f(t)}{t} \ dt \ , \quad 0 < x < 1.$$

Prove that g is also integrable.

Solution: Consider a function $F : (0,1) \times (0,1)$ defined by

$$F(x,t) = \begin{cases} \frac{f(t)}{t}, & \text{if } 0 < x < t < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then by Tonelli's theorem,

$$\int_0^1 \int_0^1 |F(x,t)| \, dx \, dt = \int_0^1 |f(t)| \, dt < +\infty,$$

so by Fubini's theorem, F is integrable. Applying Fubini's theorem again, we conclude that

$$\int_0^1 |g(x)| \, dx = \int_0^1 \left| \int_x^1 \frac{f(t)}{t} \ dt \right| \, dx \le \int_0^1 \int_0^1 |F(x,t)| \, dx \, dt < +\infty.$$

Problem 2: Let $f : [0,1] \to \mathbb{R}$ be a positive function such that f and $1/f$ are integrable. Prove that $\log f$ is integrable and

$$\lim_{q \to \infty} q \cdot \left(\int_0^1 f(x)^{1/q} \ dx - 1 \right) \ = \ \int_0^1 \log f(x) \ dx \ .$$

Solution: For any $y > 0$, $\log y < y$. Hence, $|\log f| \le \max(f, 1/f) \le f + 1/f$ which implies that $\log f$ is integrable.
Consider a function $\phi : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(t,y) = \frac{e^{ty} - 1}{t}.$$

If $y \ge 0$, then using the Taylor expansion we get

$$\phi(t,y) = \sum_{n=1}^{\infty} \frac{1}{n!} t^{n-1} y^n,$$

and so ϕ is an increasing function of t. Therefore,

$$\phi(t,y) \le \phi(1,y) = e^y - 1 \quad \text{for any } t \in (0,1).$$

Applying this with $t = 1/q,\ y = \log f(x)$, we obtain

$$q\left(f(x)^{1/q} - 1\right) \le f(x) - 1 \quad \text{whenever} f(x) > 1 \text{ and } q > 1.$$

On the other hand, if $y < 0$, then

$$|\phi(t, y)| \le -\phi(t, y)e^{-ty} = \frac{e^{-ty} - 1}{t},$$

and the previous argument yields

$$q\left|f(x)^{1/q} - 1\right| \le \frac{1}{f(x)} - 1 \quad \text{whenever } 0 < f(x) < 1 \text{ and } q > 1.$$

Therefore, the functions $F_q(x) = q\left(f(x)^{1/q} - 1\right)$ satisfy the inequality

$$|F_q(x)| \le f(x) + \frac{1}{f(x)} - 1 \quad \text{for any } q > 1,$$

where the right-hand side is an integable function. L'Hopital's rule implies

$$\lim_{q\to\infty} F_q(x) = \lim_{q\to\infty} f(x)^{1/q} \cdot \log f(x) = \log f(x)$$

for any $x \in (0, 1)$. Thus, the result follows from the Lebesgue Dominated Convergence Theorem.

Problem 3: Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Let $\mathcal{C} \subset \mathcal{A}$ be a sub-sigma algebra of $\mathcal{A}$. Prove that for any $f \in L^1(\mu)$ there exists a $\mathcal{C}-$ measurable integrable function g such that

$$\int_E g\ d\mu \ = \ \int_E f\ d\mu \quad \text{for any } E \in \mathcal{C}\ .$$

Solution: Define a function $\nu : \mathcal{C} \to \mathbb{C}$ by

$$\nu(E) = \int_E f\, d\mu.$$

Since $f \in L^1(\mu)$, ν is a complex measure on $\mathcal{C}$ and $\nu \ll \mu$. Let g be the Radon-Nikodym derivative of ν with respect to μ: $g = \frac{d\nu}{d\mu}$. The existence, $\mathcal{C}$-measurability, and integrability of g are guaranteed by the Lebesgue-Radon-Nikodym theorem. Then g satisfies the equality above.

Problem 4: Let $f_n : [0, 1] \to \mathbb{R},\ n = 1, 2, \ldots,$ be a sequence of non-negative Lebesgue measurable functions such that $\lim_{n\to\infty} f_n(x) = 0$ for almost every $x \in [0, 1]$. Prove there exists an infinite subsequence $f_{n_k},\ k = 1, 2, \ldots,$ such that the series

$$\sum_{k=1}^{\infty} f_{n_k}(x) \quad \text{converges for almost every } x \in [0, 1]\ .$$

Hint: Use Egorov's theorem.

Solution: By Egorov's theorem, for any $k \in \mathbb{N}$ there exists a set E_k with

$$m(E_k) > 1 - \frac{1}{k}$$

such that $f_n \to 0$ uniformly on E_k. Passing if necessary from E_k to $\tilde{E}_k = \bigcup_{j=1}^{k} E_j$, we may assume that $E_1 \subset E_2 \subset \dots$ Using induction, we can construct an increasing sequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that

$$|f_m(x)| \leq 2^{-k} \quad \text{for any } m \geq n_k \text{ and } x \in E_k.$$

Fix $l \in \mathbb{N}$. Since the sets E_k are nested, any $x \in E_l$ satisfies $|f_{n_k}(x)| \leq 2^{-k}$ for all $k \geq l$. Therefore,

$$\sum_{k=1}^{\infty} |f_{n_k}(x)| \leq \sum_{k=1}^{n_l - 1} |f_{n_k}(x)| + \sum_{k=n_l}^{\infty} 2^{-k} < \infty.$$

By continuity of the Lebesgue measure,

$$m\left([0,1] \setminus \bigcup_{l=1}^{\infty} E_l\right) = 0.$$

The result follows.

Problem 5: Suppose for $n = 1, 2, \dots$, the functions $F_n : [a, b] \to \mathbb{R}$ are increasing and nonnegative, and that the function F with domain $[a, b]$ defined by

$$F(x) \;=\; \sum_{n=1}^{\infty} F_n(x) \;,$$

is finite for all $x \in [a, b]$. Prove that the derivative $F'(x)$ exists a.e. and

$$F'(x) \;=\; \sum_{n=1}^{\infty} F_n'(x) \text{ for almost every } x \in [a, b].$$

Solution: The function F is increasing, and so a.e. differentiable.

To prove the equality, define $G(x) = \lim_{h \to 0+} F(x+h)$ and $G_n(x) = \lim_{h \to 0+} F_n(x+h)$. Then the functions G and G_n are increasing and right-continuous. Since F is increasing, it has only countably many points of discontinuity, and if x is a point of continuity of F and G is differentiable at x, then F is differentiable at x as well with $F'(x) = G'(x)$. Hence, it is enough to prove that

$$G'(x) = \sum_{n=1}^{\infty} G_n'(x) \quad \text{for almost all } x \in [a, b].$$

The function G defines a Lebesgue-Sieltjes measure μ_G on $[a, b]$ by

$$\mu_G((c, d]) = G(d) - G(c) \quad \text{for any } (c, d] \subset [a, b].$$

Denote the Lebesgue measure by m. We can define the Lebesgue-Stieltjes measures μ_{G_n} in a similar way. Let

$$\mu_G = \lambda + \eta, \quad \lambda \ll m, \ \eta \perp m$$

be the Lebesgue decomposition of μ_G into the absolutely continuous and the singular part, and let $d\lambda = g\,dx$, i.e., $g = \frac{d\mu}{dm}$ and $g \in L^1([a,b])$. By the Lebesgue Differentiation Theorem, $G' = g$ a.e.

Similarly, let

$$\mu_n = \lambda_n + \eta_n, \quad \lambda_n \ll m, \ \eta_n \perp m, g_n = \frac{d\mu_m}{dm}, \ g_n \in L^1([a,b]).$$

Note that since μ_G and μ_{G_n} are positive measures, all the measures here are positive. Set

$$\tilde{g} = \sum_{n=1}^{\infty} g_n, \ \tilde{\lambda} = \sum_{n=1}^{\infty} \lambda_n \text{ and } \tilde{\eta} = \sum_{n=1}^{\infty} \eta_n,$$

and as before, $G'_n = g_n$ a.e. Then $\mu_G = \tilde{\lambda} + \tilde{\eta}$, and so both measures are finite.

For any Borel set $E \subset [a,b]$,

$$\tilde{\lambda}(E) = \sum_{n=1}^{\infty} \lambda_n(E) = \sum_{n=1}^{\infty} \int_E g_n(x)\,dx = \int_E \sum_{n=1}^{\infty} g_n(x)\,dx,$$

and thus

$$\tilde{g} = \frac{d\tilde{\lambda}}{dm} = \sum_{n=1}^{\infty} g_n \in L^1([a,b]).$$

Therefore, $\tilde{\lambda} \ll m$. Also, for any $n \in \mathbb{N}$, there exists a set $E_n \subset [a,b]$ with $m(E_n) = 0$ such that $\eta_n([a,b] \setminus E_n) = 0$ Hence,

$$\tilde{\eta}([a,b] \setminus \bigcup_{n=1}^{\infty} E_n) = 0 \quad \text{and } m(\bigcup_{n=1}^{\infty} E_n) = 0,$$

so $\tilde{\eta} \perp m$. Since the decomposition of a measure into the absolutely continuous and the singular part is unique,

$$\lambda = \tilde{\lambda} \quad \text{and } \eta = \tilde{\eta}.$$

Summarizing, we have

$$G' = g = \frac{d\lambda}{dm} = \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} G'_n \text{ a.e.}$$

as claimed.