Problem 1: Use contour integration to evaluate the integral
\[ \int_0^\infty \frac{\cos x}{(1 + x^2)^2} \, dx. \]

Solution: We use the calculus of residues. Let \( \gamma_R \) be the contour consisting of the line segment \([-R, R]\) on the real axis combined with the semi-circle of radius \( R \) in the upper half plane. The direction of the contour is counter clockwise. Thus
\[
\frac{1}{2\pi i} \int_{\gamma_R} \frac{e^{iz}}{(1 + z^2)^2} = \text{Res}(f(\cdot), i), \quad \text{where } f(z) = \frac{e^{iz}}{(1 + z^2)^2}.
\]
We have that
\[
f(z) = \frac{e^{iz}}{(z - i)^2(z + i)^2}, \quad e^{iz} = e^{-1} + ie^{-1}(z - i) + O((z - i)^2), \quad \frac{1}{(z + i)^2} = -\frac{1}{4} - \frac{i}{4}(z - i) + O((z - i)^2),
\]
whence we conclude that
\[
f(z) = \frac{1}{(z - i)^2} \left[ -\frac{e^{-1}}{4} - \frac{ie^{-1}}{2}(z - i) + O((z - i)^2) \right].
\]
Hence \( \text{Res}(f(\cdot), i) = -ie^{-1}/2 \). We observe that
\[
\lim_{R \to \infty} \int_{\gamma_R \cap \{\Im z = 0\}} f(z) \, dz = 2\int_0^\infty \frac{\cos x}{(1 + x^2)^2} \, dx.
\]
If we show that
\[
\lim_{R \to \infty} \int_{\gamma_R \cap \{|z|=R\}} f(z) \, dz = 0,
\]
then we have from the residue theorem that the value of the integral in the problem is \( \pi e^{-1}/2 \). Now
\[
|f(z)| \leq \frac{1}{(R - 1)^4}, \quad \text{for } |z| = R, \exists \alpha > 0, \quad \int_{\gamma_R \cap \{|z|=R\}} |f(z)| \, |dz| \leq \frac{\pi R}{(R - 1)^4}.
\]
Letting \( R \to \infty \) we conclude the integral of \( f(\cdot) \) on the semi-circle converges to 0 as \( R \to \infty \).

Problem 2: Find a conformal mapping from the quarter disc
\[
\{ z \in \mathbb{D} : z = re^{i\theta}, \ r \in (0, 1), \ \theta \in (0, \pi/2) \}\]
to the infinite strip
\[ \{ z \in \mathbb{C} : z = x + iy, \ x \in \mathbb{R}, \ y \in (0,1) \} . \]

You may write your solution as a composite of simpler maps.

**Solution:** If \( f_1(z) = z^2 \) then \( f_1 \) maps the quarter disc \( D \) to the half disk \( D_1 = \{ z : |z| < 1, \ \Re z > 0 \} \). We can map \( D_1 \) to a quadrant using a FLT by sending
\(-1\) to 0 and \(+1\) to \( \infty \). Thus we take \( f_2(z) = (1 + z)/(1 - z) \), which maps \( D_1 \) to \( D_2 = \{ z = re^{i\theta}, \ 0 < \theta < \pi/2 \} \). Next \( f_3(z) = \frac{2}{\pi} \log z \) maps \( D_2 \) to the infinite strip \( 0 < \Re z < 1 \). The conformal mapping is therefore \( f = f_3 \circ f_2 \circ f_1 \).

**Problem 3:** Suppose \( f : \mathbb{D} \to \mathbb{C} \) is a holomorphic function on the unit disk \( \mathbb{D} \) which satisfies \(|f(z)| \leq 3\) for all \(|z| < 1\), and \( f(1/2) = 2 \).

a) Show that \( f(0) \neq 0 \).
b) Extend your result in a) by showing that \( f(\cdot) \) has no zeros in the disk \(|z| < 1/8\).

**Solution:** a). We wish to use the Schwarz lemma, whence we define \( f_1(z) = f(z)/3 \), which maps the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \) to itself. Then \( f_1(1/2) = 2/3 \). Next we use conformal mappings on \( \mathbb{D} \) to construct a function \( g : \mathbb{D} \to \mathbb{D} \) from \( f \) with \( g(0) = 0 \). Hence we need FLTs, which are conformal mappings of \( \mathbb{D} \), such that \( 0 \to 1/2 \) and \( 2/3 \to 0 \). The relevant mappings are
\[ z \to h(z) = \frac{z + 1/2}{1 + z/2}, \quad w \to k(w) = \frac{w - 2/3}{1 - 2w/3} . \]

Then \( g = k \circ f_1 \circ h \). The Schwarz lemma implies that \(|g(z)| < |z|, \ z \in \mathbb{D} - \{0\}\).
Note that \( h(-1/2) = 0, \ k(0) = -2/3 \). If \( f(0) = 0 \) then \( f_1(0) = 0 \) and so \( g(-1/2) = -2/3 \), contradicting Schwarz. We conclude that \( f(0) \neq 0 \).
b) We may extend the argument to the disk by observing that \( h^{-1} \) takes the circle centered at 0 with radius \( r \) to the circle with equation \( h(z)h'(z) = r^2 \), which is given by
\[ \left( 1 - \frac{r^2}{4} \right) (x^2 + y^2) + (1 - r^2)x + \frac{1}{4} - r^2 = 0 \quad z = x + iy . \]
This circle has center \([-2(1-r^2)/(4-r^2), 0]\) and radius \( R \) satisfying
\[ R^2 = \frac{4(1-r^2)^2}{(4-r^2)^2} - \frac{1 - 4r^2}{4 - r^2} . \]
The result follows since
\[ \frac{2(1-r^2)}{(4-r^2)} + R < \frac{2}{3} \quad \text{when} \ r = \frac{1}{8} . \]

**Problem 4:** Consider a function \( f(z) \) that is analytic for \( z \neq 0 \) and such that there exists a sequence \( z_j, \ j = 1, 2, \ldots, \) such that \( f(z_j) = 0, \ j \geq 1, \) and \( \lim_{j \to \infty} z_j = 0. \)
a) Prove that \( f \) cannot have a pole at \( z = 0 \).
b) Show by explicit example that there does exist such \( f \) which has an essential singularity at \( z = 0 \).

**Solution:** a) If \( f \) has a pole at \( z = 0 \) then \( f(z) = z^{-N}g(z) \) where \( N \geq 1 \) is an integer and \( g \) is analytic on \( \mathbb{C} \) with \( g(0) \neq 0 \). Since \( g \) is continuous it follows that \( f(z_j) \neq 0 \) for \( j \) sufficiently large, a contradiction.

b) An example is
\[
f(z) = \exp \left[ \frac{1}{z} \right] - 1, \quad z_j = \frac{1}{2\pi j}, \quad j = 1, 2, \ldots
\]

**Problem 5:** Let \( U \) be a bounded connected domain in \( \mathbb{C} \) and \( f : U \to U \) a holomorphic function which satisfies \( f(z_0) = z_0 \) and \( |f'(z_0)| < 1 \) for some \( z_0 \in U \). For \( n = 1, 2, \ldots \), let \( f^{(n)} \) be the composition function defined inductively by \( f^{(1)} = f \), \( f^{(n+1)} = f^{(n)} \circ f \). Prove that \( f^{(n)} \) converges uniformly to \( z_0 \) on compact subsets of \( U \).

**Solution:** Since \( |f'(z_0)| < 1 \) and \( f' \) is continuous there exists \( r, \delta > 0 \) such that \( |f'(z)| \leq 1 - \delta \) if \( |z - z_0| \leq r \). It follows that \( |f(z) - f(z_0)| \leq (1 - \delta)|z - z_0| \) if \( z \in D(z_0, r) = \{ z : |z - z_0| < r \} \). Since \( f(z_0) = z_0 \) we conclude that \( f(D(z_0, r)) \subset D(z_0, (1 - \delta)r) \). Proceeding by induction we have further that \( f^{(n)}(D(z_0, r)) \subset D(z_0, (1 - \delta)^n r) \), \( n = 1, 2, \ldots \). Letting \( n \to \infty \), it follows that \( f^{(n)} \) converges uniformly to \( z_0 \) on compact subsets of \( D(z_0, r) \).

We extend the result to compact subsets of \( U \) by using Montel’s theorem. Since \( U \) is bounded the family of holomorphic functions \( f^{(n)} \), \( n \geq 1 \), is bounded on every compact subset of \( U \). Suppose the sequence \( f^{(n)} \), \( n \geq 1 \), does not converge uniformly to \( z_0 \) on a compact subset \( K \subset U \). Then there exists a sequence of points \( z_j \in K \), \( j = 1, 2, \ldots \), and a subsequence \( f^{(n_j)} \), \( j = 1, 2, \ldots \), of the family \( f^{(n)} \), \( n \geq 1 \), such that \( |f^{(n_j)}(z_j) - z_0| \geq \delta > 0 \) for some positive \( \delta \). By Montel’s theorem there exists a subsequence \( f^{(n_k)} \), \( k = 1, 2, \ldots \), of \( f^{(n_j)} \), \( j = 1, 2, \ldots \), which converges uniformly on all compact subsets of \( U \) to a holomorphic function \( f^{(\infty)} \). Since the sequence \( z_j \), \( j \geq 1 \), lies in the compact set \( K \) there exists a subsequence which has a limit point \( z_\infty \in K \). We claim that \( |f^{(\infty)}(z_\infty) - z_0| \geq \delta \).

This follows from the fact that the derivatives of \( f^{(n)} \) are uniformly bounded in a nbh of \( z_\infty \), which is a consequence of the Cauchy integral formula. Since we have shown that \( f^{(\infty)} \equiv z_0 \) in \( D(z_0, r) \) it follows by analytic continuation that \( f^{(\infty)} \equiv z_0 \) in \( U \), but this contradicts the inequality \( |f^{(\infty)}(z_\infty) - z_0| \geq \delta \).