Complex Analysis Qualifying Review

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The terms "holomorphic" and "analytic" are synomymous. The complex plane is denoted $\mathbb{C}$, and $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ is the unit disc.

1. Let $g_n(z)$, $n = 1, 2, \ldots$ be a sequence of entire functions having only real zeros. Suppose that $g_n(z)$ converges locally uniformly (i.e. uniformly on compact subsets) on $\mathbb{C}$ to an entire function $g(z)$, and that $g(z)$ is not identically zero. Prove that $g(z)$ only have real zeros.

Easy consequence of Hurwitz's theorem.

2. For which integers $k \geq 1$ does there exist a holomorphic function $f(z)$ defined near the origin, such that $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^k}$ for infinitely many integers $n \geq 1$.

If k is even, then we can take $f(z) = z^k$. Now suppose k is odd. We claim no such function f exists. Set $g_+(z) = z^k$ and $g_-(z) = (-z)^k$. By assumption, the function $f(z) - g_+(z)$ vanishes at $z = \frac{1}{n}$, for infinitely many $n \geq 1$; hence must have a nonisolated zero at $z = 0$, so that $f(z) = g_+(z)$ near the origin. As $f(z) - g_-(z)$ vanishes at all points $z = -\frac{1}{n}$, the same argument shows that $f(z) = g_+(z)$ in a neighborhood of the origin. Thus $z^k = (-z)^k$ near the origin, a contradiction.

3. Construct a conformal mapping f of the domain

$$\Omega = \{z \in \mathbb{C} \mid |z| < 1, |z - \tfrac{i}{2}| > \tfrac{1}{2}\}$$

onto the unit disc $\mathbb{D} = \{w \in \mathbb{C} \mid |w| < 1\}$ with $f(-\frac{i}{3}) = 0$. You may express f as a composition of simpler maps. Draw figures to illustrate each step of the construction.

We can use the composition of the following maps:

- $z \mapsto \frac{z+i}{z-i}$
- $z \mapsto -i\pi z$
- $z \mapsto e^z$
- $z \mapsto \frac{z-i}{z+i}$.

See the attached page for an illustration.

4. Let $\varphi(z)$ be a holomorphic function on the unit disc $\mathbb{D}$, and set $f(z) = z + z^2\varphi(z)$. Assume that one of the following conditions holds:

(a) $f(\mathbb{D}) \subset \mathbb{D}$;
(b) f is one-to-one on $\mathbb{D}$ and $f(\mathbb{D}) \supset \mathbb{D}$.

Prove that $\varphi(z) = 0$ for all $z \in \mathbb{D}$.

First consider (a). As $f(0) = 0$, Schwartz's Lemma gives $|f(z)| \leq |z|$, that is, $|1 + z\varphi(z)| \leq 1$ for $z \in \mathbb{D}$. Set $h(z) = 1 + z\varphi(z)$. If $h(z) \equiv 1$, then $\varphi \equiv 0$, so suppose $h(z) \not\equiv 1$. Then $h(z)$ is an open mapping at $z = 0$, which contradicts $h(0) = 1$ and $|h(z)| \leq 1$ for $z \in \mathbb{D}$. If $\varphi(z) \not\equiv 0$, then $h(z)$ is a nonconstant holomorphic map Write $\varphi(z) = z^m g(z)$, where $m \geq 0$, g is holomorphic, and $g(0) \neq 0$.

Now consider (b). In this case $g(z) = f^{-1}(z)$ is a well-defined holomorphic function on $\mathbb{D}$ and satisfies $g(\mathbb{D}) \subset \mathbb{D}$ and $g(0) = 0$. Schwartz's Lemma now gives $|g(z)| \leq |z|$ for $z \in \mathbb{D}$, which implies $|f(z)| \geq |z|$ for z near the origin. In the notation above, this gives $|h(z)| \geq 1$ for z near the origin, so since $h(0) = 1$, the open mapping theorem shows that h must be a constant map, and again $\varphi \equiv 0$.

5. Let $f(z)$ be an entire function. Assume that f takes real values on the real axis, and purely imaginary values on the line $\operatorname{Re} z = \operatorname{Im} z$ (i.e. $y = x$, where $z = x + iy$). Prove that f takes real values on the imaginary axis. Also give an example of a function satisfying the hypotheses.

We use Schwartz reflection. Let z^* be the reflection of z across the line $\operatorname{Re} z = \operatorname{Im} z$. The fact that f is purely imaginary on this line means that if is real there, and hence Schwarz reflection gives $if(z^*) = \overline{if(z)} = -i\overline{f(z)}$, and hence $f(z^*) = -\overline{f(z)}$ for any $z \in \mathbb{C}$. If now z lies on the imaginary axis, then z^* is real, so $f(z^*)$ is real, so that $f(z) = -\overline{f(z^*)}$ is also real.

As an example of a function, we can take $f(z) = z^2$.