Complex Analysis Qualifying Review

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The terms “holomorphic” and “analytic” are synonymous. The complex plane is denoted \( \mathbb{C} \), and \( \mathbb{D} := \{ z \in \mathbb{C} \mid |z| < 1 \} \) is the unit disc.

1. Let \( g_n(z), n = 1, 2, \ldots \) be a sequence of entire functions having only real zeros. Suppose that \( g_n(z) \) converges locally uniformly (i.e. uniformly on compact subsets) on \( \mathbb{C} \) to an entire function \( g(z) \), and that \( g(z) \) is not identically zero. Prove that \( g(z) \) only have real zeros.

Easy consequence of Hurwitz’s theorem.

2. For which integers \( k \geq 1 \) does there exist a holomorphic function \( f(z) \) defined near the origin, such that \( f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^k} \) for infinitely many integers \( n \geq 1 \).

If \( k \) is even, then we can take \( f(z) = z^k \). Now suppose \( k \) is odd. We claim no such function \( f \) exists. Set \( g_+(z) = z^k \) and \( g_-(z) = (-z)^k \). By assumption, the function \( f(z) - g_+(z) \) vanishes at \( z = \frac{1}{n} \), for infinitely many \( n \geq 1 \); hence must have a nonisolated zero at \( z = 0 \), so that \( f(z) = g_+(z) \) near the origin. As \( f(z) - g_-(z) \) vanishes at all points \( z = -\frac{i}{n} \), the same argument shows that \( f(z) = g_+(z) \) in a neighborhood of the origin. Thus \( z^k = (-z)^k \) near the origin, a contradiction.

3. Construct a conformal mapping \( f \) of the domain

\[
\Omega = \{ z \in \mathbb{C} \mid |z| < 1, |z - \frac{i}{2}| > \frac{1}{2} \}
\]

onto the unit disc \( \mathbb{D} = \{ w \in \mathbb{C} \mid |w| < 1 \} \) with \( f\left(-\frac{i}{3}\right) = 0 \). You may express \( f \) as a composition of simpler maps. Draw figures to illustrate each step of the construction.

We can use the composition of the following maps:

- \( z \mapsto \frac{z+i}{z-i} \)
- \( z \mapsto -i\pi z \)
- \( z \mapsto e^z \)
- \( z \mapsto \frac{z-i}{z+i} \).
4. Let \( \varphi(z) \) be a holomorphic function on the unit disc \( \mathbb{D} \), and set \( f(z) = z + z^2 \varphi(z) \). Assume that one of the following conditions holds:

(a) \( f(\mathbb{D}) \subset \mathbb{D} \);
(b) \( f \) is one-to-one on \( \mathbb{D} \) and \( f(\mathbb{D}) \supset \mathbb{D} \).

Prove that \( \varphi(z) = 0 \) for all \( z \in \mathbb{D} \).

First consider (a). As \( f(0) = 0 \), Schwartz’s Lemma gives \( |f(z)| \leq |z| \), that is, \( |1 + z\varphi(z)| \leq 1 \) for \( z \in \mathbb{D} \). Set \( h(z) = 1 + z\varphi(z) \). If \( h(z) \equiv 1 \), then \( \varphi \equiv 0 \), so suppose \( h(z) \not\equiv 1 \). Then \( h(z) \) is an open mapping at \( z = 0 \), which contradicts \( h(0) = 1 \) and \( |h(z)| \leq 1 \) for \( z \in \mathbb{D} \). If \( \varphi(z) \not\equiv 0 \), then \( h(z) \) is a nonconstant holomorphic map. Write \( \varphi(z) = z^m g(z) \), where \( m \geq 0 \), \( g \) is holomorphic, and \( g(0) \neq 0 \).

Now consider (b). In this case \( g(z) = f^{-1}(z) \) is a well-defined holomorphic function on \( \mathbb{D} \) and satisfies \( g(\mathbb{D}) \subset \mathbb{D} \) and \( g(0) = 0 \). Schwartz’s Lemma now gives \( |g(z)| \leq |z| \) for \( z \in \mathbb{D} \), which implies \( |f(z)| \geq |z| \) for \( z \) near the origin.

In the notation above, this gives \( |h(z)| \geq 1 \) for \( z \) near the origin, so since \( h(0) = 1 \), the open mapping theorem shows that \( h \) must be a constant map, and again \( \varphi \equiv 0 \).

5. Let \( f(z) \) be an entire function. Assume that \( f \) takes real values on the real axis, and purely imaginary values on the line \( \text{Re } z = \text{Im } z \) (i.e. \( y = x \), where \( z = x + iy \)). Prove that \( f \) takes real values on the imaginary axis.

Also give an example of a function satisfying the hypotheses.

We use Schwartz reflection. Let \( z^* \) be the reflection of \( z \) across the line \( \text{Re } z = \text{Im } z \). The fact that \( f \) is purely imaginary on this line means that \( if \) is real there, and hence Schwarz reflection gives \( if(z^*) = i\overline{f(z)} = -i\overline{f(z)} \), and hence \( f(z^*) = -\overline{f(z)} \) for any \( z \in \mathbb{C} \). If now \( z \) lies on the imaginary axis, then \( z^* \) is real, so \( f(z^*) \) is real, so that \( f(z) = -\overline{f(z^*)} \) is also real.

As an example of a function, we can take \( f(z) = z^2 \).