Real Analysis Qualifying Review

January 8, 2021

Notation: \( m(E) \) is the Lebesgue measure of \( E \).

1. Construct a nowhere dense measurable subset \( E \) of \([0, 1]\), so that its Lebesgue measure \( m(E) \geq 0.9 \). (A set is called nowhere dense if its closure has no interior points.)

2. a. Construct a strictly monotone function \( f : \mathbb{R} \to \mathbb{R} \), so that the discontinuities of \( f \) are exactly all the rational points, and justify your answer.

b. Is there a monotone function \( g : \mathbb{R} \to \mathbb{R} \) so that the discontinuities of \( g \) are exactly all the irrational points? Justify your answer.

3. Let \( E \subset \mathbb{R}^n \) be measurable and \( f \) an a.e. finite measurable function on \( E \). Assume that
\[
m(\{x \in E : |f(x)| \leq k\}) = 2 - \frac{1}{k + 1},
\]
for all nonnegative integers \( k \). Find all the \( p > 0 \) so that \( f \in L^p(E) \).

4. Assume that \( f \in L^2(0, \pi) \). Show that the following inequalities
\[
\int_0^\pi (f(x) - \sin x)^2 \, dx \leq \frac{4}{9}, \quad \int_0^\pi (f(x) - \cos x)^2 \, dx \leq \frac{1}{9}
\]
do not hold simultaneously.

5. Let \( f \in L^1(\mathbb{R}) \) and \( g \) a bounded, continuous and integrable function on \( \mathbb{R} \). And let
\[
F(x) = \int_{\mathbb{R}} f(y)g(xy) \, dy.
\]
Show that \( F \) is a continuous function on \( \mathbb{R} \), and
\[
\lim_{x \to \pm \infty} F(x) = 0.
\]
(Hint: check that for any \( a, b \in \mathbb{R} \), \( \lim_{x \to \pm \infty} \int_a^b g(xy) \, dy = 0 \).)