Real Analysis Qualifying Review

.

January 8, 2021

Notation: $m(E)$ is the Lebesgue measure of E.

1. Construct a nowhere dense measurable subset E of $[0,1]$, so that its Lebesgue measure $m(E) \geq 0.9$. (A set is called nowhere dense if its closure has no interior points.)

2. a. Construct a strictly monotone function $f : \mathbb{R} \to \mathbb{R}$, so that the discontinuities of f are exactly all the rational points, and justify your answer.

b. Is there a monotone function $g : \mathbb{R} \to \mathbb{R}$ so that the discontinuities of g are exactly all the irrational points? Justify your answer.

3. Let $E \subset \mathbb{R}^n$ be measurable and f an a.e. finite measurable function on E. Assume that

$$m(\{x \in E : |f(x)| \leq k\}) = 2 - \frac{1}{k+1},$$

for all nonnegative integers k. Find all the $p > 0$ so that $f \in L^p(E)$.

4. Assume that $f \in L^2(0,\pi)$. Show that the following inequalities

$$\int_0^\pi (f(x) - \sin x)^2 \, dx \leq \frac{4}{9}, \qquad \int_0^\pi (f(x) - \cos x)^2 \, dx \leq \frac{1}{9}$$

do not hold simultaneously.

5. Let $f \in L^1(\mathbb{R})$ and g a bounded, continuous and integrable function on $\mathbb{R}$. And let

$$F(x) = \int_{\mathbb{R}} f(y) g(xy) \, dy.$$

Show that F is a continuous function on $\mathbb{R}$, and

$$\lim_{x \to \pm\infty} F(x) = 0.$$

(Hint: check that for any $a, b \in \mathbb{R}$, $\lim_{x \to \pm\infty} \int_a^b g(xy) \, dy = 0$.)