Department of Mathematics, University of Michigan
Real Analysis Qualifying Exam, August 20, 2021
Solutions

Problem 1: Let $f \in L_1((0,\infty)\times\mathbb{R})$. Define the sequence $g_n : (0,\infty) \to \mathbb{R}$ by

$$g_n(x) = \int_0^\infty e^{-\lambda} f(n\lambda, x)\, d\lambda, \quad n \in \mathbb{N}.$$

(1) Show that $\lim_{n\to\infty} \|g_n\|_1 = 0$.
(2) Show that $g_n \to 0$ a.e.

Solution: Using the change of variables formula, we get

$$g_n(x) = \frac{1}{n}\int_0^\infty e^{-t/n} f(t,x)\, dt.$$

(1) Since $f \in L_1((0,\infty)\times\mathbb{R})$, Tonelli-Fubini Theorem yields

$$\|g_n\|_1 \le \frac{1}{n}\int_{\mathbb{R}}\int_0^\infty e^{-t/n}|f(t,x)|\, dt\, dx \le \frac{1}{n}\|f\|_{L_1((0,\infty)\times\mathbb{R})} \to 0.$$

(2) For $k \in \mathbb{N}$, denote

$$E_k = \{x \in \mathbb{R} : \int_0^\infty |f(t,x)|\, dt \le k\}.$$

By the Tonelli-Fubini Theorem, E_k is measurable, and the calculation above shows that for any $x \in E_k$, $|g_n(x)| \le \frac{k}{n} \to 0$. Hence, $g_n(x) \to 0$ for all $x \in \bigcup_{k\in\mathbb{N}} E_k$, and

$$m\left(\mathbb{R} \setminus \bigcup_{k\in\mathbb{N}} E_k\right) = 0$$

again by the Tonelli-Fubini Theorem.

Problem 2: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f \in L_1(\mu)$.
Prove that $\lim_{n\to\infty}\int_X |f|^{1/n}\, d\mu$ exists and find it (the limit can be $+\infty$).

Solution: Denote $A = \{x \in X : |f(x)| > 0\}$ and $B = \{x \in X : |f(x)| \ge 1\}$. For any $x \in X$, $|f(x)|^{1/n} \to \mathbb{1}_A(x)$. For any $x \in B$, and any $n \in \mathbb{N}$, $|f(x)|^{1/n} \le |f(x)|$, and $f \in L_1(\mu)$. Hence the Lebesgue Dominated Convergence Theorem yields

$$\lim_{n\to\infty}\int_B |f|^{1/n}\, d\mu = \int_B \mathbb{1}_A\, d\mu = \mu(B).$$

On the other hand, for any $x \in B^c$, the sequence $|f(x)|^{1/n}$ is increasing. By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_{B^c} |f|^{1/n}\, d\mu = \int_{B^c} \mathbb{1}_A \, d\mu = \mu(B^c \cap A).$$

Finally,

$$\lim_{n\to\infty} \int_X |f|^{1/n}\, d\mu = \mu(B) + \mu(B^c \cap A) = \mu(A).$$

Problem 3: Let $K = \{f : [0,\infty] \to [0,\infty) : \int_0^\infty f^4\, dx \le 1\}$. Evaluate

$$\sup_{f\in K} \int_0^\infty f^3(x) e^{-x}\, dx.$$

Solution: By Hölder's inequality applied with $p = \frac{4}{3}$, for any $f \in K$,

$$\int_0^\infty f^3(x)e^{-x}\, dx \le \left(\int f^4(x)\, dx \right)^{3/4} \cdot \left(\int_0^\infty (e^{-x})^4\, dx \right)^{1/4} \le \left(\frac{1}{4}\right)^{1/4}.$$

To show that the supremum is attained, take a function f for which Hölder's inequality becomes an equality, i.e.,

$$f(x) = \sqrt{2} e^{-x}.$$

Then $f \in K$ and

$$\int_0^\infty f^3(x)e^{-x}\, dx = \left(\frac{1}{4}\right)^{1/4}.$$

Problem 4: Let $\left\{f_n : [0,1] \to \{-1,1\}\right\}_{n=1}^\infty$ be a sequence of measurable functions defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in (\frac{2k}{2^n}, \frac{2k+1}{2^n}], \quad k = 0,1,\dots,2^{n-1}-1; \\ \\ -1 & \text{if } x \in (\frac{2k+1}{2^n}, \frac{2k+2}{2^n}], \; k = 0,1,\dots,2^{n-1}-1. \end{cases}$$

Prove that

$$\int_0^1 f_n g\, dx \to 0$$

for any $g \in L_1([0,1])$.

Solution: Let $X \subset L_1([0,1])$ be any subset dense in $L_1([0,1])$. It is enough to check that for any $h \in X$, $\int_0^1 f_n h\, dx \to 0$. Indeed, assume that this condition is satisfied, and let $g \in L_1([0,1])$. Take any $\varepsilon > 0$ and let $h \in X$ be such that $\|g-h\|_1 < \varepsilon$. Then

$$\limsup \left| \int_0^1 f_n g \, dx \right| \le \limsup \left| \int_0^1 f_n h \, dx \right| + \limsup \left| \int_0^1 f_n (g-h) \, dx \right| \le \|g-h\|_1 < \varepsilon,$$

so $\int_0^1 f_n g \, dx \to 0$.

The set X can be chosen in many different ways. We show two of them below.

(1) $X = \operatorname{span}\{\mathbb{1}_{[a,b]}, \ 0 \le a < b \le 1\}$. It is enough to check the condition for $h = \mathbb{1}_{[a,b]}$. In this case

$$\int_0^1 f_n h \, dx = \int_a^b f_n \, dx \to 0$$

as required.

(2) $X = C([0,1])$. Take any $h \in X$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall x, y \in [0,1] \ |x-y| < \delta \quad \Rightarrow \quad |h(x) - h(y)| < \varepsilon.$$

Assume that $2^{-n+1} < \delta$. Then for any $k = 0, 1, \ldots, 2^{n-1} - 1$,

$$\left| \int_{\frac{2k}{2^n}}^{\frac{2k+2}{2^n}} f_n(x) h(x) \, dx \right| < \frac{\varepsilon}{2^{n-1}},$$

and so

$$\left| \int_0^1 f_n(x) h(x) \, dx \right| < \varepsilon.$$

Problem 5: Let $A \subset [0,1]$ be a set such that $m(A) \ge 0.999$ (m stands for the Lebesgue measure). Prove that there exists a point $x \in (0,1)$ such that

$$m(A \cap (x-r, x+r)) \ge r \quad \text{for any } r \in (0, 1/4).$$

Hint: use the Hardy-Littlewood Maximal Theorem. Recall that for $n = 1$, it holds with constant $C \le 4$.

Solution: Denote by Mf the maximal function of the function f:

$$Mf(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(x)| \, dx.$$

Let

$$E = \left\{ x \in \mathbb{R} : \ M\mathbb{1}_{[0,1]\setminus A}(x) > \frac{1}{2} \right\}.$$

By the Hardy-Littlewood Maximal Theorem,

$$m(E) \le 4 \cdot \frac{\left\| \mathbb{1}_{[0,1]\setminus A} \right\|_1}{1/2} \le 0.008.$$

Hence, there is a point $x \in (\frac{1}{4}, \frac{3}{4})$ such that $x \notin E$. Since $(x - \frac{1}{4}, x + \frac{1}{4}) \subset (0, 1)$, this means that for any $r \in (0, 1/4)$,

$$m(A \cap (x-r, x+r)) \ge 2r - m((x-r, x+r) \cap ([0,1] \setminus A)) \ge 2r - 2rM\mathbb{1}_{[0,1]\setminus A}(x) \ge r.$$