Department of Mathematics, University of Michigan
Analysis Qualifying Exam, August 19, 2021
Morning Session, 9.00 AM-12.00

Problem 1: Let $\mathbb{D} = \{|z| < 1\}$ be the unit disk in $\mathbb{C}$ and $f : \mathbb{D} \to \mathbb{C}$ an analytic function, which extends to a continuous function on the closure $\bar{\mathbb{D}}$ of $\mathbb{D}$.

(a) Show that for any $z_0 \in \mathbb{D}$ there is a constant C, independent of $f(\cdot)$, such that the derivative $f'(\cdot)$ satisfies $|f'(z_0)| \le C \sup_{|z|=1} |f(z)|$.

(b) Give an example to show that C may not be chosen independent of $z_0 \in \mathbb{D}$.

Solution: (a) From the Cauchy formula and the continuity of the function $f(\cdot)$ at $\partial\mathbb{D}$ we have that

$$f'(z_0) \ = \ \frac{1}{2\pi i} \int_{C(z_0,|1-z_0|)} \frac{f(z)\ dz}{(z-z_0)^2} \ ,$$

where $C(w,r)$ is the circle with center w and radius r. We conclude that

$$|f'(z_0)| \ \le \ \frac{\sup_{|z|<1} |f(z)|}{|z-z_0|} \ = \ \frac{\sup_{|z|=1} |f(z)|}{|z-z_0|} \ ,$$

where we have applied the maximum principle for the analytic function $f(\cdot)$.
(b) Take $f(z) = z^N$, whence

$$|f'(z_0)| \ = \ N|z_0|^{N-1} \sup_{|z|=1} |f(z)| \ .$$

Letting $|z_0| \to 1$ we conclude that if C is independent of z_0 then $C \ge N$ for all integers $N \ge 1$.

Problem 2: Use contour integration to find the value of the integral

$$\int_0^\infty \frac{dx}{1+x^{2021}}.$$

Solution: Let

$$\alpha = \frac{\pi}{2021}.$$

Let $R > 1$, and consider the contour γ_R consisting of the segment $[0, R]$ followed by the arc $\{Re^{it},\ t \in [0, 2\alpha]\}$ followed by the segment $\{te^{2\alpha i},\ t \in [R, 0]\}$. The contour γ_R contains a single simple pole of $f(z) = \frac{1}{1+z^{2021}}$, namely $z_0 = e^{\alpha i}$. Hence,

$$\int_{\gamma_R} \frac{dz}{1+z^{2021}} = 2\pi i \cdot \operatorname{Res}(f, z_0) = 2\pi i \cdot \frac{1}{2021 z_0^{2020}} = -2\pi i \cdot \frac{1}{2021 e^{-\alpha i}} = -2\alpha i e^{\alpha i}.$$

Letting $R \to \infty$, we obtain

$$\left| \int_{\{te^{2\alpha i},\ t\in[R,0]\}} \frac{dz}{1+z^{2021}} \right| \le \frac{2\alpha R}{R^{2021}-1} \to 0,$$

and the sum of the integrals over the line segments converges to

$$(1-e^{2\alpha i}) \int_0^\infty \frac{dx}{1+x^{2021}}.$$

Therefore,

$$\int_0^\infty \frac{dx}{1+x^{2021}} = -\frac{2\alpha i e^{\alpha i}}{1-e^{2\alpha i}} = \frac{\alpha}{\sin\alpha}.$$

Problem 3: Let $z_n,\ n = 1, 2, \dots$, be a sequence in $\mathbb{C}$ such that $\lim_{n\to\infty} |z_n| = \infty$ and $a_n,\ n = 1, 2, \dots$, a sequence in $\mathbb{C} - \{0\}$ satisfying

$$\sum_{n=1}^{\infty} \frac{|a_n|}{|z_n|} \ < \ \infty \ .$$

(a) Show that the function $f(\cdot)$ defined by

$$f(z) \ = \ \sum_{n=1}^{\infty} \frac{a_n}{z-z_n} \ , \quad z \in \mathbb{C} \ ,$$

is a meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with simple poles at the points $z_n,\ n = 1, 2, \dots$.

(b) Does $f(\cdot)$ extend to a meromorphic function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$? Explain your answer.

Solution: (a) For $N = 1, 2, \dots$, define f_N by

$$f_N(z) \ = \ \sum_{n=1}^{N} \frac{a_n}{z-z_n} \ ,$$

so f_N is a rational function with simple poles at $z_1, \dots, z_N$. Let $R > 0$ and N be chosen so that $|z_n| \ge 2R$ if $n > N$. Then

$$|f_N(z) - f(z)| \ \le \ 2 \sum_{n=N+1}^{\infty} \frac{|a_n|}{|z_n|} \quad \text{if } |z| < R \ .$$

By Casorati-Weierstrass theorem $f(\cdot)$ is analytic in $\mathbb{C} \setminus \{z_n;\ n = 1, 2, \dots\}$. Furthermore if $|z_n| < R$ then $\lim_{z\to z_n}(z-z_n)f(z) = \lim_{z\to z_n}(z-z_n)f_N(z) = a_n$. Hence $f(\cdot)$ has a simple pole at z_n with residue $a_n \ne 0$.

(b) The only meromorphic functions $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are rational functions, which have a finite number of singularities. Since $f(\cdot)$ has an infinite number of singularities, it cannot extend to a meromorphic function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Problem 4: Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(1) = 1$ and $|f(z)| = 1$ if $|z| = 1$.

(a) Show that

$$\overline{f(z)} f\left(\frac{1}{\bar{z}}\right) = 1 \quad \text{for } z \in \mathbb{C} .$$

(b) Conclude that $f(z) = z^n$ for some integer $n \geq 0$.

Solution: (a) We define the function $g(\cdot)$ with domain $\mathbb{C} \setminus \{0\}$ by

$$g(z) = 1 \Big/ \overline{f\left(\frac{1}{\bar{z}}\right)} , \quad |z| > 0 .$$

Then $g(\cdot)$ is analytic and $g(z) = f(z)$ when $|z| = 1$. Since the zeros of the function $z \to g(z) - f(z)$ are not isolated we have $g(\cdot) \equiv f(\cdot)$ in $\mathbb{C} \setminus \{0\}$.
(b) Suppose $f(0) \neq 0$. Then (a) implies $f(\cdot)$ is bounded on $\mathbb{C}$, whence $f(1) = 1$ and Liouville's theorem implies $f(\cdot) \equiv 1$. Otherwise $f(\cdot)$ has a zero of order $n \geq 1$ at 0. We then replace $f(\cdot)$ by the function $g(z) = f(z)/z^n$ in the previous argument to conclude that $f(z) = z^n$.

Problem 5:

(a) Show that $\mathbb{D}$ is not conformally equivalent to $\mathbb{C}$.
(b) Find an analytic mapping $f : \mathbb{D} \to \mathbb{C}$ such that $f(\mathbb{D}) = \mathbb{C}$.

Solution: (a) Conformal equivalence implies there is an analytic function $f : \mathbb{C} \to \mathbb{D}$, whence $\sup_{z \in \mathbb{C}} |f(z)| \leq 1$. The result follows from Liouville's theorem.
(b) We first conformally map $\mathbb{D}$ onto the right half plane by means of the transformation

$$w(z) = \frac{1 - z}{1 + z} .$$

The mapping $w \to (w - 1)^2$ maps the right half plane onto $\mathbb{C}$.