Problem 1: Let \( \mathbb{D} = \{ |z| < 1 \} \) be the unit disk in \( \mathbb{C} \) and \( f : \mathbb{D} \to \mathbb{C} \) an analytic function, which extends to a continuous function on the closure \( \bar{\mathbb{D}} \) of \( \mathbb{D} \).

(a) Show that for any \( z_0 \in \mathbb{D} \) there is a constant \( C \), independent of \( f(\cdot) \), such that the derivative \( f'(\cdot) \) satisfies \( |f'(z_0)| \leq C \sup_{|z|=1} |f(z)| \).

(b) Give an example to show that \( C \) may not be chosen independent of \( z_0 \in \mathbb{D} \).

Solution: (a) From the Cauchy formula and the continuity of the function \( f(\cdot) \) at \( \partial \mathbb{D} \) we have that
\[
f'(z_0) = \frac{1}{2\pi i} \int_{C(z_0,|1-z_0|)} \frac{f(z) \, dz}{(z-z_0)^2},
\]
where \( C(w,r) \) is the circle with center \( w \) and radius \( r \). We conclude that
\[
|f'(z_0)| \leq \sup_{|z|<1} |f(z)| = \sup_{|z|=1} |f(z)|, 
\]
where we have applied the maximum principle for the analytic function \( f(\cdot) \).

(b) Take \( f(z) = z^N \), whence
\[
|f'(z_0)| = N|z_0|^{N-1} \sup_{|z|=1} |f(z)|.
\]
Letting \( |z_0| \to 1 \) we conclude that if \( C \) is independent of \( z_0 \) then \( C \geq N \) for all integers \( N \geq 1 \).

Problem 2: Use contour integration to find the value of the integral
\[
\int_0^\infty \frac{dx}{1+x^{2021}}.
\]

Solution: Let
\[
\alpha = \frac{\pi}{2021}.
\]
Let \( R > 1 \), and consider the contour \( \gamma_R \) consisting of the segment \([0,R]\) followed by the arc \( \{Re^{it}, t \in [0,2\alpha]\} \) followed by the segment \( \{te^{2\alpha i}, t \in [R,0]\} \). The contour \( \gamma_R \) contains a single simple pole of \( f(z) = \frac{1}{1+z^{2021}} \), namely \( z_0 = e^{\alpha i} \). Hence,
\[
\int_{\gamma_R} \frac{dz}{1+z^{2021}} = 2\pi i \cdot \text{Res}(f, z_0) = 2\pi i \cdot \frac{1}{2021z_0^{2020}} = -2\pi i \cdot \frac{1}{2021e^{-\alpha i}} = -2\alpha i e^{\alpha i}.
\]
Letting $R \to \infty$, we obtain

$$\left| \int_{\{e^{2\alpha i}, t \in [R,0]\}} \frac{dz}{1 + z^{2021}} \right| \leq \frac{2\alpha R}{R^{2021} - 1} \to 0,$$

and the sum of the integrals over the line segments converges to

$$(1 - e^{2\alpha i}) \int_{0}^{\infty} \frac{dx}{1 + x^{2021}}.$$ 

Therefore,

$$\int_{0}^{\infty} \frac{dx}{1 + x^{2021}} = -\frac{2\alpha ie^{\alpha i}}{1 - e^{2\alpha i}} = \frac{\alpha}{\sin \alpha}.$$ 

**Problem 3:** Let $z_n, n = 1, 2, \ldots$, be a sequence in $\mathbb{C}$ such that $\lim_{n \to \infty} |z_n| = \infty$ and $a_n, n = 1, 2, \ldots$, a sequence in $\mathbb{C} - \{0\}$ satisfying

$$\sum_{n=1}^{\infty} \frac{|a_n|}{|z_n|} < \infty.$$ 

(a) Show that the function $f(\cdot)$ defined by

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n}{z - z_n}, \quad z \in \mathbb{C},$$ 

is a meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with simple poles at the points $z_n, n = 1, 2, \ldots$. 

(b) Does $f(\cdot)$ extend to a meromorphic function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$? Explain your answer.

**Solution:** (a) For $N = 1, 2, \ldots$, define $f_N$ by

$$f_N(z) = \sum_{n=1}^{N} \frac{a_n}{z - z_n},$$

so $f_N$ is a rational function with simple poles at $z_1, \ldots, z_N$. Let $R > 0$ and $N$ be chosen so that $|z_n| \geq 2R$ if $n > N$. Then

$$|f_N(z) - f(z)| \leq 2 \sum_{n=N+1}^{\infty} \frac{|a_n|}{|z_n|} \quad \text{if} \quad |z| < R.$$

By Casorati-Weierstrass theorem $f(\cdot)$ is analytic in $\mathbb{C} \setminus \{z_n; n = 1, 2, \ldots\}$. Furthermore if $|z_n| < R$ then $\lim_{z \to z_n} (z - z_n)f(z) = \lim_{z \to z_n} (z - z_n)f_N(z) = a_n$. Hence $f(\cdot)$ has a simple pole at $z_n$ with residue $a_n \neq 0$.

(b) The only meromorphic functions $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are rational functions, which have a finite number of singularities. Since $f(\cdot)$ has an infinite number of singularities, it cannot extend to a meromorphic function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. 

Problem 4: Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function such that \( f(1) = 1 \) and \( |f(z)| = 1 \) if \( |z| = 1 \).

(a) Show that
\[
f(z) f\left(\frac{1}{z}\right) = 1 \quad \text{for } z \in \mathbb{C}.
\]

(b) Conclude that \( f(z) = z^n \) for some integer \( n \geq 0 \).

Solution: (a) We define the function \( g(\cdot) \) with domain \( \mathbb{C} \setminus \{0\} \) by
\[
g(z) = \frac{1}{f\left(\frac{1}{z}\right)}, \quad |z| > 0.
\]
Then \( g(\cdot) \) is analytic and \( g(z) = f(z) \) when \( |z| = 1 \). Since the zeros of the function \( z \to g(z) - f(z) \) are not isolated we have \( g(\cdot) \equiv f(\cdot) \) in \( \mathbb{C} \setminus \{0\} \).

(b) Suppose \( f(0) \neq 0 \). Then (a) implies \( f(\cdot) \) is bounded on \( \mathbb{C} \), whence \( f(1) = 1 \) and Liouville’s theorem implies \( f(\cdot) \equiv 1 \). Otherwise \( f(\cdot) \) has a zero of order \( n \geq 1 \) at \( 0 \). We then replace \( f(\cdot) \) by the function \( g(z) = f(z)/z^n \) in the previous argument to conclude that \( f(z) = z^n \).

Problem 5:

(a) Show that \( \mathbb{D} \) is not conformally equivalent to \( \mathbb{C} \).

(b) Find an analytic mapping \( f : \mathbb{D} \to \mathbb{C} \) such that \( f(\mathbb{D}) = \mathbb{C} \).

Solution: (a) Conformal equivalence implies there is an analytic function \( f : \mathbb{C} \to \mathbb{D} \), whence \( \sup_{z \in \mathbb{C}} |f(z)| \leq 1 \). The result follows from Liouville’s theorem.

(b) We first conformally map \( \mathbb{D} \) onto the right half plane by means of the transformation
\[
w(z) = \frac{1 - z}{1 + z}.
\]
The mapping \( w \to (w - 1)^2 \) maps the right half plane onto \( \mathbb{C} \).