Department of Mathematics, University of Michigan
Real Analysis Qualifying Exam, August 20, 2021
Morning Session, 9.00 AM-12.00 AM

Problem 1: Let $f \in L_1((0,\infty)\times\mathbb{R})$. Define the sequence $g_n : (0,\infty) \to \mathbb{R}$ by

$$g_n(x) = \int_0^\infty e^{-\lambda} f(n\lambda, x)\, d\lambda, \quad n \in \mathbb{N}.$$

(1) Show that $\lim_{n\to\infty} \|g_n\|_1 = 0$.
(2) Show that $g_n \to 0$ a.e.

Problem 2: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f \in L_1(\mu)$.
Prove that $\lim_{n\to\infty} \int_X |f|^{1/n}\, d\mu$ exists and find it (the limit can be $+\infty$).

Problem 3: Let $K = \{f : [0,\infty] \to [0,\infty) : \int_0^\infty f^4\, dx \le 1\}$. Evaluate

$$\sup_{f\in K} \int_0^\infty f^3(x) e^{-x}\, dx.$$

Problem 4: Let $\left\{f_n : [0,1] \to \{-1,1\}\right\}_{n=1}^\infty$ be a sequence of measurable functions defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in (\frac{2k}{2^n}, \frac{2k+1}{2^n}], \quad k = 0,1,\ldots,2^{n-1}-1; \\ \\ -1 & \text{if } x \in (\frac{2k+1}{2^n}, \frac{2k+2}{2^n}],\ k = 0,1,\ldots,2^{n-1}-1. \end{cases}$$

Prove that

$$\int_0^1 f_n g\, dx \to 0$$

for any $g \in L_1([0,1])$.

Problem 5: Let $A \subset [0,1]$ be a set such that $m(A) \ge 0.999$ (m stands for the Lebesgue measure). Prove that there exists a point $x \in (0,1)$ such that

$$m(A \cap (x-r, x+r)) \ge r \quad \text{for any } r \in (0, 1/4).$$

Hint: use the Hardy-Littlewood Maximal Theorem. Recall that for $n = 1$, it holds with constant $C \le 4$.