Department of Mathematics, University of Michigan
Analysis Qualifying Exam, January 4, 2020
Morning Session, 9.00 AM-12.00

Problem 1: Let $P(z)$, $z \in \mathbb{C}$, be the polynomial

$$P(z) \; = \; z^n + a_{n-1}z^{n-1} + \cdots + a_0 \; .$$

The coefficients $a_j \in \mathbb{C}$, $j = 0, .., n-1$ can be arbitrary.
Prove that $\sup_{|z|=1} |P(z)| \geq 1$.

Problem 2: Use contour integration to evaluate the integral

$$\int_0^\infty \frac{\ln x}{1+x^2} \, dx \; .$$

Problem 3: Let $f \in L^1([0,1])$ and $\Omega = \mathbb{C} \backslash [0,1]$. Prove that the function

$$F(z) \; = \; \int_0^1 \frac{f(t)}{t-z} \, dt$$

is holomorphic on Ω.

Problem 4: Let $\Omega = \{z \in \mathbb{C} : |z - 1/3| > 1/3 \text{ and } |z-1| < 1\}$.

(a) Find a conformal mapping f from Ω onto the open unit disk $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$.

(b) Show that we may choose $f(\cdot)$ to be of the form $f(z) = \tan g(z)$ with $g(1) = 0$, $g'(1) > 0$, and obtain a formula for the function $g(\cdot)$.

Problem 5: Let f be a holomorphic function in $\mathbb{C}$ satisfying $f(0) = 0$, $f(1) = 1$, and $|f(z)| \leq |z|$ for all $|z| \leq 1$. Show that:

(a) $f'(1) \in \mathbb{R}$. Hint: Consider the function $\theta \mapsto |f(e^{i\theta})|^2$.

(b) $f'(1) \geq 1$. Hint: Consider $\mathrm{Re}\left(\dfrac{f(x)-1}{x-1}\right)$ for $x \in (0,1)$.

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Afternoon Session, 2.00-5.00 PM

Problem 1: Let $E \subset \mathbb{R}$ be a bounded Lebesgue measurable set. Show there is a measurable subset F of E such that $m(F) = m(E)/2$.

Problem 2: For a closed interval $I \subset \mathbb{R}$, a function $f : I \to \mathbb{R}$ is convex if for all $x, y \in I$ and $0 < \lambda < 1$ one has $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$.

(a) Prove that the convex function f is continuous and of bounded variation.

(b) Give an example to show that f is not necessarily absolutely continuous.

Problem 3: Let $(X, \mathcal{A}, \mu)$ be a measure space, and $f_n : X \to (0, \infty)$, $n = 1, 2, ...,$ be a sequence of measurable functions such that

$$\sum_{n=1}^{\infty} \mu(f_n^{-1}(1/n, \infty)) \ < \ \infty \ .$$

Prove that $f_n \to 0$ a.e.

Problem 4: Let f_n, $n = 1, 2, ...,$ be a bounded sequence in $L^2([0,1])$ such that f_n converges almost everywhere to a function f.

(a) Prove that the function $x \to f(x)/x^{1/3}$ is integrable.

(b) Prove further that

$$\lim_{n\to\infty} \int_0^1 \frac{f_n(x)}{x^{1/3}} \ dx \ = \ \int_0^1 \frac{f(x)}{x^{1/3}} \ dx \ .$$

Problem 5: Suppose f and g are non-negative real-valued measurable functions on $(0, 1)$ which satisfy

$$\int_0^1 f(x) \ dx = 3, \qquad \int_0^1 g(x) \ dx = 2 \ .$$

Let $E = \{x \in (0,1) \ : \ f(x) > g(x)\}$.

(a) Prove that $m(E) > 0$, and give an example to show that $m(E)$ may be arbitrarily small.

(b) Suppose that $\int_0^1 f(x)^3 \ dx = 64$. Prove that $m(E) \ge 1/8$.