1. a. Let $A = \{x \in E, \ |f(x)| \neq 0\}$

WTS $m(A) = 0$

Let $A_n = \{x \in E \cap B(0,n) \ | \ |f(x)| > \frac{1}{n}\}$

$\bigcup A_n = A$. it suffices to show $m(A_n) = 0 \ \forall n$

Let $\phi(x) = \begin{cases} 1 & x \in A_n, \ f(x) > 0 \\ -1 & x \in A_n, \ f(x) < 0 \\ 0 & \text{else where} \end{cases}$

$\Rightarrow \phi$ is a compactly supported simple fcn

by assumption $\int f(x)\phi(x) = \int_{A_n} |f(x)| \, dx = 0$

$\Rightarrow \frac{1}{n} m(A_n) = 0 \Rightarrow m(A_n) = 0$ ✓

b. Let $1 < p' < \infty$ be s.t. $\frac{1}{p'} + \frac{1}{p} = 1$

We know compactly supported cont. fcn is dense in $L^{p'}(E)$. i.e. $\forall \ g \in L^{q'}(E)$

$\exists \ g_n$, compactly supported cont. on $\mathbb{R}^n$

s.t $\|g_n - g\|_{L^{q'}} \to 0$ as $n \to \infty$

$\because \int f g_n = 0$ and by Hölder :

$$\left| \int f (g_n - g)\, dx \right| \le \|f\|_p \, \|g_n - g\|_{p'} \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \int f g \, dx = 0 \qquad \forall \; g \in L^{p'}(E)$$

$$\Rightarrow \quad f = 0 \;\; \text{a.e. by part a.}$$

2. Let $F_n(x) = \sum_{k=1}^{n} |f_k(x)|$

$F_n(x)$ is an increasing seq of nonneg. measurable fcns

and by Monotone Conv. Thm

$$\lim_{n\to\infty} \int_E F_n(x)\, dx \quad \text{exist \&} = \int \lim_{n\to\infty} F_n(x)\, dx$$

$$\Rightarrow \int \lim_{n\to\infty} F_n(x)\, dx = \sum_{k=1}^{\infty} \int |f_k(x)|\, dx < \infty$$

So $\lim_{n\to\infty} F_n(x)$ is a.e. finite & $\in L^1(E)$

Let $A = \{ x \in E \mid \lim_{n\to\infty} F_n(x) \text{ is finite} \}$

So $m(E \setminus A) = 0$

and $\forall \; x \in A$

$$\left| \sum_{k=m}^{n} f_k(x) \right| \le \sum_{k=m}^{n} |f_k(x)| \to 0 \quad \text{as } n, m \to \infty$$

So $\forall \; x \in A$ $\left\{ \sum_{k=1}^{n} f_k(x) \right\}$ is Cauchy so

$\sum_{k=1}^{\infty} f_k(x)$ converges $\forall\ x \in A \Rightarrow$ part a.

let $f(x) = \sum_{k=1}^{\infty} f_k(x)$

f is measurable on E. and

$$\left| \sum_{k=1}^{n} f_k(x) \right| \le \sum_{k=1}^{n} |f_k(x)| \le \sum_{k=1}^{\infty} |f_k(x)| := F$$

$F \in L^1(E)$ by part a.

by Dominated Conv. Thm. $f \in L^1(E)$ and

$$\lim_{n\to\infty} \int_E \sum_{k=1}^{n} f_k(x)\, dx = \int_E f(x)\, dx \quad \Rightarrow \text{part b.}$$

3. let $\mathcal{O} \subset \mathbb{R}$ by and open set

we know $\exists$ a seq of mutually disjoint intervals (a_n, b_n) s.t. $\mathcal{O} = \bigcup_n (a_n, b_n)$

So $f^{-1}(\mathcal{O}) = \bigcup_n f^{-1}(a_n, b_n)$

$$= \bigcup_n f^{-1}(a_n, \infty) \cap f^{-1}(-\infty, b_n)$$

It suffices to write $f^{-1}(a, \infty)$ as a F_σ set. $\forall a$.

$\because \lim_{n\to\infty} f_n(x) = f(x)$

$$f^{-1}(a,\infty) = \{x : f(x) > a\} = \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} \bigcup_{n=\ell}^{\infty} \{x : f_n(x) \geq a + \tfrac{1}{k}\}$$

$\{x : f_n(x) \geq a + \frac{1}{k}\}$ is closed $\because$ f_n is cont.

$\Rightarrow$ $f^{-1}(a,\infty)$ is F_σ. //

4) $f \in C^1[0,1]$, so $\forall x \in \mathbb{R}$

$\frac{f(y)}{x-y}$ is a measurable fcn of y.

a)

WT find all $x \in \mathbb{R}$ s.t. $\left|\frac{f(y)}{x-y}\right| \in L^1([0,1])$

① if $x_0 \notin [0,1]$, then $\left|\frac{1}{x_0-y}\right| \leq \frac{1}{\operatorname{dist}(x_0,[0,1])} < \infty$

$\Rightarrow$ $\left|\frac{f(y)}{x_0-y}\right|$ is a bounded measurable fcn. so $\in L^1([0,1])$

② if $x_0 \in [0,1]$ & $f(x_0) = 0$ by MVT,

$$\left|\frac{f(y)}{x_0-y}\right| = \left|\frac{f(y)-f(x_0)}{x_0-y}\right| \leq \max_{[0,1]} |f'(y)| < \infty$$

$\left|\frac{f(y)}{x_0-y}\right|$ is bdd measurable on $[0,1]$. so $\in L^1([0,1])$

③ if $x_0 \in [0,1]$, $f(x_0) \neq 0$ $\Rightarrow$ $\exists\ \delta > 0$, s.t.

$\forall\ y \in (x_0-\delta, x_0+\delta) \cap [0,1] =: B$ $|f(y)| \geq \frac{1}{2}|f(x_0)| > 0$

$\because$ $\frac{1}{x_0-y} \notin L^1([0,1])$ $\Rightarrow$ $\left|\frac{f(y)}{x_0-y}\right| \notin L^1(B)$ $\Rightarrow$ $\frac{f(y)}{x_0-y} \notin L^1([0,1])$

$\Rightarrow$ $A = \{x \in \mathbb{R} \mid x \notin [0,1], \text{ or } f(x) = 0\}$

b) $F(x) = \int_0^1 \frac{f(y)}{x-y}\,dy$ is cont on A.

b). $F(x) = \int_0 \frac{}{x-y} \, dy$

pf ① if $x_0 \notin [0,1]$, then $\delta_0 = dist(x_0, [0,1]) > 0$

$\forall$ seq $x_n \to x_0$. $\exists N$ s.t. $\forall n > N$, $dist(x_n, [0,1]) > \frac{\delta_0}{2} > 0$

$$\Rightarrow \left| \frac{f(y)}{x_n - y} \right| \le \frac{2|f(y)|}{\delta_0} \in L^1([0,1]) \quad \forall n > N$$

by D.C.T. $F(x_n) \to F(x_0)$ so F cont at x_0

② if $x_0 \in (0,1)$. and $x_n \in A$, s.t. $x_n \to x_0$.

$\Rightarrow \exists N$. s.t. $\forall n > N$, $x_n \in (0,1) \Rightarrow f(x_n) = 0 \ \forall n > N$

$$\Rightarrow \left| \frac{f(y)}{x_n - y} \right| = \left| \frac{f(y) - f(x_n)}{x_n - y} \right| \le M := \max_{[0,1]} \{ |f'(y)| \} \quad \forall n > N$$

& $\frac{f(y)}{x_n - y} \to \frac{f(y)}{x_0 - y} \quad \forall y \in [0,1] \setminus \{x_0\}$ as $n \to \infty$

D.C.T $\Rightarrow$ $F(x_n) \to F(x_0)$. so F cont at x_0

③ if $x_0 = 0$, $x_n \in A$, $x_n \to 0$, $\Rightarrow$ either $x_n \in [0,1]$. or $x_n \notin [0,1]$. only need to show for the case that $x_n \notin [0,1]$;

assume $x_n < 0$, $x_n \to 0$. $\Rightarrow$

$$\frac{f(y)}{x_n - y} \to \frac{f(y)}{0 - y} \quad \forall y \in (0,1]$$

and $$\left| \frac{f(y)}{x_n - y} \right| = \left| \frac{f(y) - f(0)}{x_n - y} \right| \le M \frac{|y|}{|x_n - y|} \le M \quad \forall y \in [0,1]$$

by D. C. T. $$\int_0^1 \frac{f(y)}{x_n - y} dy \to \int_0^1 \frac{f(y)}{0 - y} dy.$$

5.

$$\int \left(\frac{f(x)-f(y)}{x-y}\right)^4 dy = \frac{4}{3}\int \left(\frac{f(x)-f(y)}{x-y}\right)^3 f'(y)\,dy$$

$$\leq \frac{4}{3}\left(\int \left|\left(\frac{f(x)-f(y)}{x-y}\right)^3\right|^{\frac{4}{3}} dy\right)^{3/4}\left(\int |f'(y)|^4 dy\right)^{\frac{1}{4}}$$

$$= \frac{4}{3}\left(\int \left|\frac{f(x)-f(y)}{x-y}\right|^4 dy\right)^{3/4}\left(\int |f'(y)|^4 dy\right)^{\frac{1}{4}}$$

$$\Rightarrow \left(\int \left|\frac{f(x)-f(y)}{x-y}\right|^4 dy\right)^{1/4} \leq \frac{4}{3}\left(\int |f'(y)|^4 dy\right)^{\frac{1}{4}}$$

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