Department of Mathematics, University of Michigan
Analysis Qualifying Exam, September 2, 2019
Morning Session, 9.00 AM-12.00

Problem 1: Let $\mathbb{D} = \{z \ : \ |z| < 1\}$, and let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function such that $f(0) = f'(0) = \cdots = f^{(n)}(0) = 0$. Prove that the inequality $|f(z)| \le |z|^{n+1}$ holds for all $z \in \mathbb{D}$.

Problem 2: Define the function f on $\mathbb{C}$ by $f(z) = \sin\left(\frac{1}{\sin z}\right)$.

(a) Show that f is holomorphic in the region $\{z \in \mathbb{C} \ : \ 0 < |z| < \pi\}$.

(b) Determine (with justification) if the point $z = 0$ is a removable singularity, a pole or an essential singularity.

Problem 3: Let $\alpha_1, \alpha_2, .., \alpha_n$ be the zeros, repeated according to multiplicity, of a polynomial

$$p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n \ , \quad c_0 \neq 0, \quad c_n \neq 0 \ .$$

Show using contour integration that

$$\sum_{k=1}^{n} \frac{1}{\alpha_k} \ = \ -\frac{p'(0)}{p(0)} \ ,$$

and obtain a similar formula for

$$\sum_{k=1}^{n} \frac{1}{\alpha_k^2} \ .$$

Problem 4: Prove that the function

$$f(z) \ = \ \prod_{n=1}^{\infty} \frac{n}{z} \sin\left(\frac{z}{n}\right)$$

is well-defined and holomorphic in $\mathbb{C} \setminus \{0\}$.

Problem 5: Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the curve $\gamma(t) = te^{it}$, $0 < t < 2\pi$. Evaluate the integral

$$\int_\gamma \frac{dz}{1 + z^2}$$

Hint: Note that γ is not a closed curve.

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Afternoon Session, 2.00-5.00 PM

Problem 1: Let E be the set of all $x \in (0,1)$ such that there exists a sequence of irreducible fractions $\{p_n/q_n\}_{n\in\mathbb{N}}$ with $p_n, q_n \in \mathbb{N}$, $q_1 < q_2 < \cdots$ such that

$$\left| x - \frac{p_n}{q_n} \right| \le \frac{1}{q_n^3}, \quad n = 1, 2, \ldots$$

Prove that the Lebesgue measure of E is zero.

Problem 2: Let f be a measurable function on $(0,\infty)$, and for $n = 1, 2, ..,$ let f_n be defined by

$$f_n(x) = f(x)e^{-x}\left[1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right].$$

Suppose $f \in L^2[(0,\infty)]$. Prove that $\lim_{n\to\infty} \|f_n - f\|_{L^2[(0,\infty)]} = 0$

Problem 3: Let $f : \mathbb{R} \times (0,1) \to \mathbb{R}$ be a measurable function such that for any $y \in (0,1)$,

$$\int_{\mathbb{R}} f^2(x,y)\, dx \le 1.$$

Prove there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$, with $\lim_{n\to\infty} x_n = +\infty$, such that

$$\lim_{n\to\infty} \int_0^1 |f(x_n, y)|\, dy = 0.$$

Problem 4: Let (X, Ω, μ) be a measure space with $\mu(X) = 1$, and let $f \in L^2(\mu)$ be a non-negative function satisfying $\int_X f\, d\mu \ge 1$. Prove that

$$\mu(\{x \in X : f(x) > 1\}) \ge \frac{\left(\int_X f\, d\mu - 1\right)^2}{\int_X f^2\, d\mu}.$$

Problem 5: A function $f : (0,1) \to \mathbb{R}$ is locally Lipschitz if for any $x \in (0,1)$ there is an open interval I_x with $x \in I_x \subset (0,1)$ and a constant C_x such that $|f(y) - f(y')| \le C_x|y - y'|$ for $y, y' \in I_x$.

(a) Prove that a locally Lipschitz function $f(\cdot)$ is absolutely continuous on any compact subinterval $[a,b] \subset (0,1)$.

(b) Give an example of a locally Lipschitz function $f : (0,1) \to \mathbb{R}$ which extends to a continuous function on the closed interval $[0,1]$, but is not absolutely continuous on $[0,1]$.