Department of Mathematics, University of Michigan
Analysis Qualifying Exam, January 8, 2019
Morning Session, 9.00 AM-noon

Problem 1: Find all entire functions f with the property that $|f|$ is harmonic.

Problem 2:

(a) Find a quadratic function $Q(z)$ such that

$$5 - 4\cos\theta \;=\; Q(z)/z \quad \text{for } z = e^{i\theta},\ \theta \in \mathbb{R}\,.$$

(b) Use your representation in (a) and contour integration to compute the value of the integral

$$\int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta}\,.$$

Problem 3: Let $\mathcal{F}$ denote the set of functions that are analytic on a neighborhood of the closed unit disk $|z| \le 1$. Find

$$\sup\{|f(0)| : f \in \mathcal{F} \text{ with } f(1/2) = 0 = f(1/3) \text{ and } |f(z)| \le 1 \text{ when } |z| = 1\}\,.$$

Is the supremum attained?

Hint: The function $\frac{z-1/2}{1-z/2}\frac{z-1/3}{1-z/3}$ is useful.

Problem 4: Let Ω be a bounded open subset of $\mathbb{C}$ whose boundary $b\Omega$ is a C^1 simple closed curve and let p be a monic polynomial of degree n with distinct roots $w_1, \dots, w_n$ all contained in Ω. Let f be an analytic function on a neighborhood of $\overline{\Omega}$. Let q be the entire function defined by

$$q(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{p(\zeta) - p(z)}{p(\zeta)}\frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

(a) Show that q is a polynomial of degree $\le n - 1$.

(b) Evaluate $q(w_j)$ for $j = 1, \dots, n$.

Problem 5: Let F be a finite subset of $\mathbb{C}$ and let $h : \mathbb{C} \setminus F \to \mathbb{C} \setminus F$ be an analytic bijection.

(a) Suppose $z_j \in \mathbb{C} \setminus F$, $z_j \to z_* \in F \cup \{\infty\}$, $h(z_j) \to w_*$. Show that $w_* \in F \cup \{\infty\}$.

(b) Show that h must be a rational function.

(c) Must h be a linear fractional transformation?

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Afternoon Session, 2.00-5.00 PM

Note: Lebesgue measure is assumed throughout.

Problem 1: Let $f_n : (0,1) \to \mathbb{R}$, $n = 1, 2, \ldots$, be a sequence of measurable functions and consider the sequence $g_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$, of functions defined by

$$g_n(x) = \frac{f_1(x) + \cdots + f_n(x)}{n}.$$

(a) Suppose that there is a function $f : (0,1) \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all $x \in (0,1)$. Show that $\lim_{n\to\infty} g_n(x) = f(x)$ for almost all $x \in (0,1)$.

(b) Let $f_n(x) = \sin \pi n x$, $0 < x < 1$. Show that $\lim_{n\to\infty} g_n(x) = 0$ for almost all $x \in (0,1)$, but also that for almost all $x \in (0,1)$ the limit $\lim_{n\to\infty} f_n(x)$ fails to exist.

Problem 2: Let p satisfy $1 < p < \infty$ and $f : (0,\infty) \to \mathbb{R}$ be in $L^p(0,\infty)$. Prove that

$$\int_0^\infty \left(\int_0^1 |f(sx)|\, ds \right)^p dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p \, dx .$$

Hint: You may quote Minkowski's integral inequality

$$\left(\int_T \left| \int_Y h(t,y)\, dy \right|^p dt \right)^{1/p} \le \int_Y \left(\int_T |h(t,y)|^p \, dt \right)^{1/p} dy.$$

Problem 3: Let $c : (0,\infty) \to \mathbb{R}$ be non-negative and measurable such that the function $x \mapsto (1+x)c(x)$ is integrable.

(a) Prove that the function $w : [0,\infty) \to \mathbb{R}$ defined by $w(x) = \int_x^\infty c(x')\, dx'$ is continuous and decreasing with $\lim_{x\to\infty} w(x) = 0$.

(b) Show that the function $w(\cdot)$ is integrable.

Problem 4: Let $f : [0,1] \to \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = x^2 \sin(x^{-2})$ for $0 < x \le 1$. Is f of bounded variation?

Problem 5: Let $\phi : \mathbb{R} \to \mathbb{R}$ be an integrable function with integral equal to 1. For $N = 1, 2, \ldots$, define the functions $\phi_N : \mathbb{R} \to \mathbb{R}$ by $\phi_N(x) = N\phi(Nx)$, $x \in \mathbb{R}$. Prove that the Fourier transforms

$$\hat{\phi}_N(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} \phi_N(x)\, dx$$

converge uniformly on any finite interval as $N \to \infty$ to the function that is identically 1 on $\mathbb{R}$. Is convergence also uniform on all of $\mathbb{R}$? Explain your answer.