N.B.: $D$ below denotes the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

(1) Suppose that we have
(a) simply-connected domains $\Omega_1, \Omega_2 \subset \mathbb{C}$;
(b) distinct points $z_1, w_1 \in \Omega_1$;
(c) distinct points $z_2, w_2 \in \Omega_2$.
Show that there is an analytic map $f : \Omega_1 \to \Omega_2$ satisfying $f(z_1) = z_2, f(w_1) = w_2$ or an analytic map $f : \Omega_2 \to \Omega_1$ satisfying $f(z_2) = z_1, f(w_2) = w_1$ (or both).

(2) Let $\Sigma$ be the strip $\{z \in \mathbb{C} : |\text{Im}(z)| < 1\}$, and let $F$ be analytic on $\Sigma$, continuous on $\Sigma$, and verifying $|F(z)| \leq 1$ on $\partial \Sigma$.
(a) Show that $|F(z)|$ is not necessarily $\leq 1$ on $\Sigma$.
(b) Show that if, in addition, $F$ verifies the hypothesis $|F(z)| \leq Ce^{b|z|^p}$, for some constants $C, b > 0$ and $0 < \rho < 2$, then $|F(z)| \leq 1$ on $\Sigma$.

Hint: Consider $F_\epsilon(z) := e^{-\epsilon z^2}F(z)$, for all $\epsilon > 0$.

(3) Let $f$ be an analytic function on $D$ which is continuous on $\overline{D}$ with $|f(z)| \equiv 1$ on $\partial D$. Show that $f$ is the restriction to $D$ of a rational function on $\mathbb{C}$.

(4) Let $D^* := D \setminus \{0\}$ be the punctured unit disk. Let $f : D^* \to \mathbb{C}$ be analytic and injective.
(a) Show that $\{f(z) : 0 < |z| < 1/2\}$ is not dense in $\mathbb{C}$.
(b) Show that $f$ has a meromorphic extension to $D$. (Do not quote Picard’s theorem here.)

(5) Suppose that $g, h$ are continuous, $\mathbb{C}$-valued and nowhere vanishing functions on $\{z \in \mathbb{C} : |z| < 2\}, \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, respectively. Suppose that $f = g/h$ is analytic on the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.
(a) Show that there are continuous, single-valued functions $\log g$ on $\{z \in \mathbb{C} : |z| < 2\}$, and $\log h$ on $\{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$.

(continued over)
(b) Show that $U = \log g - \log h$ is analytic on the annulus $A$.

(c) Show that $f$ can be written as $f(z) = G(z)/H(z)$ where $G, H$ are nowhere vanishing analytic functions on $\{z \in \mathbb{C} : |z| < 2\}, \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, respectively.
Analysis Qualifying Review
Thursday, May 3, 2018
Afternoon Session, 2:00 - 5:00 PM

N.B.: Lebesgue measure is denoted below by “m”.

(1) Let \((X, \mathcal{A}, \mu)\) be a measure space, and let \(f \geq 0\) be in \(L^1(X, \mu)\). Let a set function \(\nu\) be defined on \(\mathcal{A}\) by \(\nu(A) = \int_A f d\mu\). Show that \(\nu\) is a measure on \(\mathcal{A}\) and that for any \(\nu\)-integrable function \(g\),

\[
\int_X g d\nu = \int_X g \cdot f d\mu.
\]

(2) Provide a (detailed) proof or a (detailed) counterexample to the following statement: If \(E\) is a bounded open subset of \(\mathbb{R}\) then the boundary of \(E\) has Lebesgue measure zero.

(3) Show that \(\{f \in L^2(\mathbb{R}, m) : \int_{\mathbb{R}} |f| = \infty\}\) is dense in \(L^2(\mathbb{R}, m)\).

(4) Let \(\mu\) be a non-negative measure on the interval \((-1, 1)\) with the property that all open subintervals of \((-1, 1)\) are \(\mu\)-measurable and \(\mu((-1, 1)) = 1\). Let \(f : \mathbb{R} \to \mathbb{R}\) be uniformly continuous and let \(f_n : \mathbb{R} \to \mathbb{R}\) be the function defined by \(f_n(x) = \int_{-1}^1 f(x + \frac{t}{n}) \, d\mu(t)\).

(a) Show that each \(f_n\) is uniformly continuous.

(b) Show that the \(f_n\) converge uniformly to \(f\).

(5) Let \(f_n\) be a sequence of functions in \(L^\infty([0, 1], m)\) satisfying the conditions

\(\text{(i)}\) \(\|f_n\|_{L^\infty([0, 1], m)} \leq 1,\) and
\n\(\text{(ii)}\) \(\int_{[a,b]} f_n \, dm \to 0\) for all \(0 \leq a < b \leq 1.\)

(a.) Show that \(\int_{[0,1]} f_n g \, dm \to 0\) for all \(g \in L^1([0, 1], m).\)

(b.) Under assumptions (i) and (ii), does \(f_n \to 0\) in \(L^1([0, 1], m)\)?