Analysis Qualifying Review. May 4, 2017

Morning Session, 9:00 am - 12:00 pm

1. Let $(f_j)_{j=1}^{\infty}$ be a sequence of measurable functions on a measure space $(X, \mathcal{M}, \mu)$. Suppose that the series
$$\sum_{j=1}^{\infty} \mu\{x \in X \mid |f_j(x)| \geq \epsilon\}$$
converges for every $\epsilon > 0$. Prove that $f_j(x) \to 0$ almost everywhere on X.

2. Let $E \subset [0,1]$ be the middle-third Cantor set, i.e. $E = [0,1] \setminus \bigcup_{n=1}^{\infty} U_n$, where $U_1 = (1/3, 2/3)$, $U_2 = (1/9, 2/9) \cup (7/9, 8/9)$ etc. Find a function $f \in C^{\infty}(\mathbb{R})$ such that $f \geq 0$ and $\{x \in \mathbb{R} \mid f(x) = 0\} = E$.

3. Let $\alpha < 1$. Prove the existence of the limit
$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{1/n} e^{\alpha x}\, dx,$$
and calculate it

4. Let $\beta > 1$ and $C > 0$. Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \leq C|x-y|^{\beta}$ for all $x, y \in \mathbb{R}$.

5. Construct a function $f \in L^1(\mathbb{R}^n)$ such that $f \notin L^p(U)$ for any open subset $U \subset \mathbb{R}^n$ and any $p > 1$.

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Afternoon Session, 2:00 pm - 5:00 pm

1. Let $f(z)$ be an entire function such that $f(0) = 1 + \pi i$ and $\operatorname{Re} f(z) \geq 1$ when $|z| < 1$. Compute $f'(0)$.

2. Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disc and $a \in \mathbb{D} \setminus \{0\}$ a point. Find all analytic functions $f(z)$ on $\mathbb{D}$ such that
   - $|f(z)| < 1$ for all $z \in \mathbb{D}$;
   - $f(a) = 0$ and $f(0) = a$.

3. Use residues to compute the integral $\int_0^\infty \frac{sintx}{x} dx$ for any $t \in \mathbb{R}$. Show all your steps.

4. Prove that for any real number $a > 1$, the equation $ze^{a-z} = 1$ has exactly one solution in the unit disc, and that this solution is real and positive.

5. Let $f(z)$ be a complex-valued C^∞ function defined on a connected open subset Ω of the complex plane. Assume that $f(z)$ and $f^2(z)$ are both harmonic (i.e. the real and imaginary parts of these functions are harmonic). Prove that either $f(z)$ or $\overline{f(z)}$ is analytic in Ω.