Analysis Qualifying Review. January 7, 2016

Morning Session, 9:00 am - 12:00 pm

1. (a) Let f_n be a sequence of continuous real-valued functions on $[0,1]$ which converges uniformly to f. Prove that $\lim_{n\to\infty} f_n(x_n) = f(1/2)$ for any sequence $\{x_n\}$ that converges to $1/2$.

   (b) Suppose the convergence $f_n \to f$ is only pointwise. Does the conclusion still hold? Explain.

2. Show that
$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \tag{1}$$
for $-\pi \le x \le \pi$.

3. Let R be the unit square $[0,1] \times [0,1]$ in the plane, and let μ be the usual Lebesgue measure on the real Cartesian plane. Let N be the function that assigns to each real number x in the unit interval the positive integer that indicates the first place in the decimal expansion of x after the decimal point where the first 0 occurs. If there are two expansions, use the expansion that ends in a string of zeroes. If 0 does not occur, let $N(x) = \infty$. For example, $N(0.0) = 1$, $N(0.5) = 2$, $N(1/9) = \infty$, and $N(0.4763014\dots) = 5$. Evaluate $\iint_R y^{-N(x)}\, d\mu$.

4. Let $(f_n)_1^\infty$ be a sequence in $L^p(\mu)$, where $1 \le p < \infty$. Show that if $\lim ||f_n - f||_p = 0$, where $f \in L^p(\mu)$, then (f_n) converges to f in measure.

5. Suppose that $f \in L^p([-1,1])$ for all $1 \le p < \infty$. Prove that the integral
$$\int_{-1}^{1} \frac{|f(x)|}{|x|^s}\, dx$$
is finite for all $0 < s < 1$.

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Afternoon Session, 2:00 pm - 5:00 pm

1. Let $f(z)$ and $g(z)$ be entire functions for which there exists a constant $C > 0$ such that $|f(z)| \leq C|g(z)|$ for all z. Prove that there exists a constant c such that $f(z) = cg(z)$ for all z.

2. Find a conformal mapping $w = f(z)$ that takes the first quadrant in the z-plane onto the unit disc in the w-plane, and such that $f(0) = 1$, $f(1+i) = 0$.

3. Find all analytic functions on the unit disc that satisfy $f'(\frac{1}{n}) = f(\frac{1}{n})$ for $n = 2, 3, 4, \ldots$. Justify your answer.

4. Let $a \in \mathbb{C}$ with $|a| \neq 1$. Evaluate the integral
$$\oint_{|z|=1} \frac{\overline{z}}{a - z^{100}}\, dz.$$

5. Let $f(z)$ be an analytic function in the unit disc $\{|z| < 1\}$. Prove that there exists a sequence $(z_n)_1^\infty$ in the disc such that $\lim_{n\to\infty} |z_n| = 1$ and such that $\sup_n |f(z_n)| < \infty$.