1. (a) Let \( f_n \) be a sequence of continuous real-valued functions on \([0, 1]\) which converges uniformly to \( f \). Prove that \( \lim_{n \to \infty} f_n(x_n) = f(1/2) \) for any sequence \( \{x_n\} \) that converges to 1/2.

(b) Suppose the convergence \( f_n \to f \) is only pointwise. Does the conclusion still hold? Explain.

2. Show that
\[
x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx
\]
for \(-\pi \leq x \leq \pi\).

3. Let \( R \) be the unit square \([0, 1] \times [0, 1]\) in the plane, and let \( \mu \) be the usual Lebesgue measure on the real Cartesian plane. Let \( N \) be the function that assigns to each real number \( x \) in the unit interval the positive integer that indicates the first place in the decimal expansion of \( x \) after the decimal point where the first 0 occurs. If there are two expansions, use the expansion that ends in a string of zeroes. If 0 does not occur, let \( N(x) = \infty \). For example, \( N(0.0) = 1 \), \( N(0.5) = 2 \), \( N(1/9) = \infty \), and 
\( N(0.4763014\ldots) = 5 \). Evaluate \( \iint_R y^{-N(x)} \, d\mu \).

4. Let \( (f_n)_{n=1}^{\infty} \) be a sequence in \( L^p(\mu) \), where \( 1 \leq p < \infty \). Show that if \( \lim ||f_n - f||_p = 0 \), where \( f \in L^p(\mu) \), then \( (f_n) \) converges to \( f \) in measure.

5. Suppose that \( f \in L^p([-1, 1]) \) for all \( 1 \leq p < \infty \). Prove that the integral
\[
\int_{-1}^{1} \frac{|f(x)|}{|x|^s} \, dx
\]
is finite for all \( 0 < s < 1 \).
1. Let $f(z)$ and $g(z)$ be entire functions for which there exists a constant $C > 0$ such that $|f(z)| \leq C|g(z)|$ for all $z$. Prove that there exists a constant $c$ such that $f(z) = cg(z)$ for all $z$.

2. Find a conformal mapping $w = f(z)$ that takes the first quadrant in the $z$-plane onto the unit disc in the $w$-plane, and such that $f(0) = 1$, $f(1 + i) = 0$.

3. Find all analytic functions on the unit disc that satisfy $f'(\frac{1}{n}) = f(\frac{1}{n})$ for $n = 2, 3, 4, \ldots$. Justify your answer.

4. Let $a \in \mathbb{C}$ with $|a| \neq 1$. Evaluate the integral

$$\oint_{|z|=1} \frac{\bar{z}}{a - z^{100}} \, dz.$$ 

5. Let $f(z)$ be an analytic function in the unit disc $\{|z| < 1\}$. Prove that there exists a sequence $(z_n)_{n=1}^{\infty}$ in the disc such that $\lim_{n \to \infty} |z_n| = 1$ and such that $\sup_n |f(z_n)| < \infty$. 
