Analysis Qualifying Review. January 7, 2017

Morning Session, 9:00 am - 12:00 pm

1. (a) Let f_n be a sequence of continuous real-valued functions on $[0, 1]$ which converges uniformly to f. Prove that $\lim_{n\to\infty} f_n(x_n) = f(1/2)$ for any sequence $\{x_n\}$ that converges to $1/2$.

   (b) Suppose the convergence $f_n \to f$ is only pointwise. Does the conclusion still hold? Explain.

   Solution

   (a) Fix $\epsilon > 0$ and let $N_0 \in \mathbb{N}$ be such that $n \geq N_0$ implies $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in [0, 1]$.
   Since the convergence is uniform, f is continuous, so we can pick $\delta > 0$ such that $|f(x) - f(1/2)| < \epsilon/2$ for all $x \in [0, 1]$ with $|x - 1/2| < \delta$. Let $N_1 \in \mathbb{N}$ be such that $n \geq N_1$ implies $|x_n - 1/2| < \delta$. Then $n \geq \max\{N_0, N_1\}$ implies $|f_n(x_n) - f(1/2)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(1/2)| < \epsilon/2 + \epsilon/2 = \epsilon$.

   (b) The conclusion is false, as the following counterexample shows: Define

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1/2 - 1/2n \\ 2nx - (n-1) & \text{if } 1/2 - 1/2n \leq x \leq 1/2 \\ 1 & \text{if } 1/2 \leq x \leq 1 \end{cases} \tag{1}$$

   Let $x_n = 1/2 - 1/n$. Then $x_n \to 1/2$ but $f(1/2) = 1 \neq 0 = \lim_n f_n(x_n)$.

2. Show that

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \tag{2}$$

   for $-\pi \leq x \leq \pi$.

   Solution: Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ of period 2π and defined by $f(x) = x^2$ for $-\pi \leq x \leq \pi$. Its Fourier series converges uniformly since f is Lipschitz continuous (for example). Now the nth Fourier coefficient is

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx} dx$$

   A direct calculation shows that $\hat{f}(0) = \pi^2/3$ and $\hat{f}(n) = \frac{2(-1)^n}{n^2}$ for $n \neq 0$. Hence

$$f(x) = \sum_{-\infty}^{\infty} \hat{f}(n) e^{inx},$$

which yields the desired formula since $e^{inx} + e^{-inx} = 2\cos nx$.

3. Let R be the unit square $[0,1] \times [0,1]$ in the plane, and let μ be the usual Lebesgue measure on the real Cartesian plane. Let N be the function that assigns to each real number x in the unit interval the positive integer that indicates the first place in the decimal expansion of x after the decimal point where the first 0 occurs. If there are two expansions, use the expansion that ends in a string of zeroes. If 0 does not occur, let $N(x) = \infty$. For example, $N(0.0) = 1$, $N(0.5) = 2$, $N(1/9) = \infty$, and $N(0.4763014\ldots) = 5$. Evaluate $\iint_R y^{-N(x)}\, d\mu$.

   Solution: The interval $[0,1]$ consists of one interval of length $\frac{1}{10}$ where $N = 1$, 9 intervals of length 10^{-2} where $N = 2$, and, in general, 9^{k-1} intervals of length 10^{-k} where $N = k$. It follows that for y fixed,

$$\begin{aligned} \int_0^1 y^{-N(x)}\, dx &= \sum_{k=1}^{\infty} 9^k 10^{-k} y^{-k} \\ &= \frac{y}{10} \frac{1}{1 - \frac{9}{10y}} \\ &= \frac{y}{10 - 9y}. \end{aligned}$$

   Hence, by the Fubini-Tonelli theorem,

$$\iint_R y^{-N(x)}\, d\mu = \int_0^1 \frac{y}{10 - 9y}\, dy = \int_0^1 \left(-\frac{1}{9} + \frac{10}{81} \frac{1}{\frac{10}{9} - y} \right) dy = \frac{10}{81} \log 10 - \frac{1}{9}.$$

4. Let $(f_n)_1^\infty$ be a sequence in $L^p(\mu)$, where $1 \le p < \infty$. Show that if $\lim ||f_n - f||_p = 0$, where $f \in L^p(\mu)$, then (f_n) converges to f in measure.

   Solution: Pick $\epsilon > 0$ and consider the measurable set

$$E_{n,\epsilon} := \{x \in X \ |f_n(x) - f(x)| \ge \epsilon\}.$$

   for $n \ge 1$. Then

$$\int |f_n - f|^p d\mu \ge \int \chi_{E_{n,\epsilon}} |f_n - f|^p d\mu \ge \epsilon^p \int \chi_{E_{n,\epsilon}} d\mu = \epsilon^p \mu(E_{n,\epsilon}). \tag{3}$$

   Since the left hand side tends to zero as $n \to \infty$, we see that $\lim_{n\to\infty} \mu(E_{n,\epsilon}) = 0$ for every $\epsilon > 0$, which precisely means that $f_n \to f$ in measure.

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Afternoon Session, 2:00 pm - 5:00 pm

1. Let $f(z)$ and $g(z)$ be entire functions for which there exists a constant $C > 0$ such that $|f(z)| \leq C|g(z)|$ for all z. Prove that there exists a constant c such that $f(z) = cg(z)$ for all z.

   Solution: If g is identically zero then so is f and any c will do. Otherwise, g has isolated zeros. The function $h(z) = f(z)/g(z)$ is holomorphic outside the zeros of g and satisfies $|h(z)| \leq C$. It suffices to prove that h extends to an entire function, since then h will be a bounded entire function, and hence constant by Liouville's Theorem. Now consider any zero z_0 of g. We can write $f(z) = (z - z_0)^m \tilde{f}(z)$ and $g(z) = (z - z_0)^n \tilde{g}(z)$, where m and n are nonnegative integers and where $\tilde{f}$, $\tilde{g}$ are holomorphic near z_0 (in fact, they are entire) and do not vanish at z_0. The estimate $|f(z)| \leq C|g(z)|$ implies that $m \geq n$. Thus $h(z) = (z - z_0)^{m-n} \tilde{f}(z)/\tilde{g}(z)$ in a punctured neighborhood of z_0, and the right-hand side is a holomorphic function in a neighborhood of z_0. Thus $h(z)$ extends to a holomorphic function in a neighborhood of z_0. This completes the proof since z_0 was an arbitrary zero of g.

2. Find a conformal mapping $w = f(z)$ that takes the first quadrant in the z-plane onto the unit disc in the w-plane, and such that $f(0) = 1$, $f(1 + i) = 0$.

   Solution: First set $\zeta = z^2$. This takes the first quadrant onto the upper half plane, $z = 0$ to $\zeta = 0$ and $z = 1 + i$ to $\zeta = 2i$. Now set $w = \frac{2i-\zeta}{\zeta+2i}$. This takes the upper half plane to the unit circle, $\zeta = 0$ to $w = 1$, and $\zeta = 2i$ to $w = 0$. Thus we can set

$$w = f(z) = \frac{2i - z^2}{z^2 + 2i}.$$

3. Find all analytic functions on the unit disc that satisfy $f'(\frac{1}{n}) = f(\frac{1}{n})$ for $n = 2, 3, 4, \ldots$. Justify your answer.

   Solution: The function $g(z) := f'(z) - f(z)$ has zeros at the points $z = \frac{1}{n}$, $n \geq 2$, and these points accumulate at the origin, so we must have $g(z) \equiv 0$, that is, $f'(z) = f(z)$. This implies $\frac{d}{dz}(e^{-z} f(z)) = 0$, so $f(z) = ce^z$ for some complex number c. Conversely, if $f(z) = ce^z$, then it is clear that $f' = f$, and in particular $f'(\frac{1}{n}) = f(\frac{1}{n})$ for $n \geq 2$.

4. Let $a \in \mathbb{C}$ with $|a| \neq 1$. Evaluate the integral

$$\oint_{|z|=1} \frac{\overline{z}}{a - z^{100}} \, dz.$$

Solution: Using $\overline{z}z = 1$ for $|z| = 1$, the integral is

$$\oint_{|z|=1} \frac{1}{z(a - z^{100})} dz = \oint_{|z|=1} f(z) dz$$

If $|a| > 1$, then f has a exactly one simple pole at $z = 0$ in $|z| < 1$. and the residue of f is $1/a$ there, so the integral is equal to $\frac{2\pi i}{a}$.

If instead $|a| < 1$, then there are 101 poles in $|z| < 1$ and no poles on $|z| > 1$. We can therefore replace the countour $|z| = 1$ by $|z| = R$, for $R \gg 1$. As $R \to \infty$, it follows from the Cachy estimates (ML bound) that the integral is zero.

5. Let $f(z)$ be an analytic function in the unit disc $\{|z| < 1\}$. Prove that there exists a sequence $(z_n)_1^\infty$ in the disc such that $\lim_{n\to\infty} |z_n| = 1$ and such that $\sup_n |f(z_n)| < \infty$.

Solution: Suppose no such sequence exists. Then f only have finitely many zeros in the disc, say at $z = a_i$, $1 \leq i \leq r$, with multiplicities m_i, $1 \leq i \leq r$. Set $p(z) = \prod_{i=1}^r (z - a_i)^{m_i}$. Then the function $h(z) = p(z)/f(z)$ is analytic on the unit disc and tends to zero at the boundary. It then follows from the Cauchy estimates (or the maximum principle) that h is identically zero, a contradiction.

5. Suppose that $f \in L^p([-1,1])$ for all $1 \le p < \infty$. Prove that the integral

$$\int_{-1}^{1} \frac{|f(x)|}{|x|^s}\, dx$$

is finite for all $0 < s < 1$.

Solution: This follows from Hölders inequality. Pick p sufficiently large so that $q = \frac{p}{p-1} < s^{-1}$. Then

$$\int_{-1}^{1} \frac{|f(x)|}{|x|^s}\, dx \le \|f\|_p \left(\int_{-1}^{1} |x|^{-qs}\, dx \right)^{1/q} < \infty.$$