Analysis Qualifying Review. September 10, 2016

Afternoon Session, 2:00 pm - 5:00 pm

1. Let $\mathbb{D}_r = \{z \in \mathbb{C} \mid |z| < r\}$ be the disc of radius r, for $r > 0$. Prove that if f is an analytic function on $\mathbb{D}_2$, then there exists a constant $C > 0$ such that $|f(z) - f(w)| \leq C|z - w|$ for all $z, w \in \mathbb{D}_1$.

2. The function $f(z) = \frac{1}{\cos \pi z}$ has a convergent Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z + i)^n.$$

Compute $\limsup_{n\to\infty} |a_n|^{1/n}$.

3. Let f be an entire function. Assume that there exist a constant $C > 0$ and an integer $n \geq 0$ such that $|f(z)| \leq C|z|^n$ for all $z \in \mathbb{C}$. Prove that f is a polynomial of degree at most n.

4. Prove that for any $r < 1$ there exists $n > 1$ such that the polynomial

$$P(z) = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots + \frac{1}{n}z^n$$

has no zero in the punctured disc $\{z \in \mathbb{C} \mid 0 < |z| < r\}$.

5. Find a conformal map from Ω onto D, where $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\Omega = \{z \in \mathbb{C} \mid |z| < 1, \mathrm{Re} z > 0, \mathrm{Im}\, z > 0\}$. You may express the map as a composition of simpler maps.

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Morning Session, 9:00 am - 12:00 noon

1. Construct an open set $U \subset [0,1]$ such that:

(a) U is dense in $[0,1]$;
(b) U has Lebesgue measure $\mu(U) < 1$;
(c) $\mu(U \cap I) > 0$ for every open interval $I \subset [0,1]$.

2. Let $A \subset \mathbb{R}$ be a measurable set. Suppose

$$\frac{\mu(A \cap I)}{\mu(I)} \leq \frac{1}{2}$$

for every finite interval $I \subset \mathbb{R}$, where μ is the Lebesgue measure on $\mathbb{R}$. Show that

$$\mu(A) = 0.$$

3. Let f be an absolutely continuous function on $\mathbb{R}$ such that $f \in L^1(\mathbb{R})$. Suppose that

$$\lim_{t \to 0^+} \int_{\mathbb{R}} \left| \frac{f(x+t) - f(x)}{t} \right| dx = 0.$$

Show that

$$f = 0.$$

4. Let (S, Σ, μ) be a measure space, and $f \in L^1(S, \Sigma, \mu)$. Prove the identity

$$\|f\|_1 = \int_0^\infty \mu\{x \in S : \, |f(x)| \geq t\} \, dt.$$

(Hint: use that $|f(x)| = \int_0^{|f(x)|} dt$.)

5. Let $1 < p < \infty$ and $f \in L^p[0, \infty)$. Show that

$$\left| \int_0^x f(t) \, dt \right| \leq \|f\|_p \, x^{1 - \frac{1}{p}}$$

for every $x > 0$.