Analysis QR exam, May 5, 2016

Morning Session, 9:00–12:00

1. Find all entire functions $f(z)$ satisfying $|f(z)| \geq e^{|z|}$ for all $z \in \mathbb{C}$.

2. Assume that $a > 0$. Evaluate the integral
$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx.$$

3. Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, where $a_i \in \mathbb{C}$ are constants. Prove that there is at least one point on $|z| = 1$ satisfying $|P(z)| \geq 1$.

4. Suppose that f is analytic on the annulus $\{z \in \mathbb{C} : 1/2 < |z| < 2\}$ except for the simple pole at 1. Suppose that the residue of f at $z = 1$ is 1. Let $\sum_{n=-\infty}^{\infty} a_n z^n$ and $\sum_{n=-\infty}^{\infty} b_n z^n$ be the Laurent expansion of f on the annuli $\{z : 1/2 < |z| < 1\}$ and $\{z : 1 < |z| < 2\}$, respectively. Compute $b_n - a_n$ for every integer n.

5. Find all functions f which is analytic in an open set containing $\overline{\mathbb{D}}$ and satisfies $|f(z)| = 1$ for $|z| = 1$. Here $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open disk of radius 1.

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Afternoon Session, 2:00–5:00

1. (a) Prove that for every Borel measurable set $E \subset [0,1]$,

$$\lim_{n\to\infty} \int_E \sin(2\pi n x)dx = 0.$$

   (b) Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions satisfying

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

   Does it imply that $\lim_{n\to\infty} f_n(x) = 0$? Prove it or give a counter-example.

2. Let $f \in L^1([0,1])$, and let g be a bounded increasing function on $[0,1]$. Assume that for any interval $[a,b] \subset [0,1]$,

$$\left| \int_a^b f(x)\, dx \right|^2 \leq (g(b) - g(a))(b-a).$$

   Prove that $f \in L^2([0,1])$.

3. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Prove that there is a finite measure ν on $(X, \mathcal{B})$ with the property that

$$\nu(E) = 0 \text{ if and only if } \mu(E) = 0.$$

4. Assume that $f \in L^1([0,\infty)) \cap C([0,\infty))$. Evaluate the limit

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^\infty f(x) e^{-\frac{x}{\epsilon}}\, dx.$$

   Prove your assertion.

5. Let f be a function on $\mathbb{R}^n$. Assume that for any $\epsilon > 0$, there exist measurable functions $g, h \in L^1(\mathbb{R}^n)$ such that $g(x) \leq f(x) \leq h(x)$ for all $x \in \mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} (h(x) - g(x))\, dx < \epsilon.$$

   Prove that f is measurable on $\mathbb{R}^n$ and $f \in L^1(\mathbb{R}^n)$.