Analysis QR exam, January 9, 2016

Morning Session, 9:00–12:00

1. Let E be the subset of the interval $[0,1]$ consisting of the points x that has a decimal expansion
$$x = 0.a_1a_2a_3a_4\cdots$$
with $a_n \neq 5$ for all $n = 1, 2, 3, \cdots$. (For example, both 0.5 and 0.6 are in E since 0.5 has an expansion $0.5 = 0.4999\cdots$ and 0.6 has an expansion $0.6 = 0.6000\cdots$.) Show that E is Lebesgue measurable and evaluate the Lebesgue measure of E.

2. The Fourier transform of a complex valued function f on $\mathbb{R}$ is defined by
$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx, \qquad \xi \in \mathbb{R}.$$
Prove that if $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is continuous on $\mathbb{R}$ and
$$\lim_{|\xi|\to\infty} \widehat{f}(\xi) = 0.$$

3. Let $p > 0$ and let E be a measurable subset of $\mathbb{R}^d$. Suppose that $f_n, f \in L^p(E)$, and $\|f_n - f\|_p \to 0$ as $n \to \infty$.
   (a) Show that for every $\epsilon > 0$,
   $$\lim_{n\to\infty} m(\{x \in E : |f_n(x) - f(x)| > \epsilon\}) = 0.$$
   Here m denotes the Lebesgue measure.
   (b) Show that there exists a subsequence f_{n_j} such that $f_{n_j}(x) \to f(x)$ for almost every $x \in E$.
   (There is a theorem states (a) implies (b). You need to prove this theorem instead of quoting the theorem.)

4. Let $f \in L^1[a,b]$. Prove that if
$$\lim_{h\to 0} \frac{1}{h} \int_a^b |f(x+h) - f(x)|\, dx = 0,$$
then there is a constant c such that $f(x) = c$ for almost every $x \in (a,b)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported C^1 function. Show that there is a constant $C > 0$, independent of f, such that for all $x \in \mathbb{R}$,
$$\int \frac{(f(x) - f(y))^4}{(x-y)^4}\, dy \le C \|f'\|^4_{L^4(\mathbb{R})}.$$
(Hint: You may like to use integration by parts.)

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Afternoon Session, 2:00–5:00

1. Prove that if an entire function f satisfies $\text{Re}(f(z)) > 0$ for all $z \in \mathbb{C}$, then f is a constant function.

2. Let f be a non-vanishing analytic function on $D = \{z : |z| < 1\}$ which is continuous on $\bar{D} = \{z : |z| \leq 1\}$. Suppose that $|f(e^{2\pi it})| = e^{t(1-t)}$, for $t \in [0, 1]$. Find $|f(0)|$.

3. Evaluate the integral
$$\int_0^\infty \frac{\ln x}{x^2 - 1}\, dx.$$

4. Let
$$f(z) = z^3 + \frac{1}{(z-1)^2}.$$
   (a) How many times does $f(z)$ wind around the origin as z moves along the circle $|z| = 2$ counterclockwise?
   (b) How many zeros, counting multiplicity, does f have inside the circle $|z| = 2$?

5. Let $D = \{z : |z| < 1\}$. Suppose f is an analytic function on $D \setminus \{0\}$ satisfying $\iint_D |f(x+iy)|^2\, dx\, dy < \infty$ where the integral is the usual $\mathbb{R}^2$-area integral. Prove that f has a removable singularity at $z = 0$.