Algebra II QR — January 2024

Problem 1. Let G be a finite simple group which contains an element of order 55. Prove that the index of any proper subgroup of G is at least 16.

Problem 2. Prove that any group of order $455 = 5 \cdot 7 \cdot 13$ is abelian.

Problem 3. Let $f(x) \in k[x]$ be an irreducible polynomial where k is a field of characteristic 0 with algebraic closure $\bar{k}$. Prove that there does not exist an element $a \in \bar{k}$ so that $f(a) = f(a+1) = 0$.

Problem 4. Let $f(x) \in F[x]$ an irreducible, separable polynomial over a field F, and let E be a splitting field for $f(x)$ over F. Prove that if $\mathrm{Gal}(E/F)$ is abelian, then for any root $a \in E$ of $f(x)$ we have $E = F(a)$.

Problem 5. Prove that $\mathbf{Q}(\sqrt{2+\sqrt{2}})$ is a Galois field extension of $\mathbf{Q}$, and compute its Galois group.

Hint: The following two facts may be useful.

(1) (Eisenstein's criterion) If $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbf{Z}[x]$ and p is a prime such that p divides all a_i but p^2 does not divide a_0, then $f(x)$ is irreducible as an element of $\mathbf{Q}[x]$.

(2) If $\alpha = \sqrt{2+\sqrt{2}}$ and $\beta = \sqrt{2-\sqrt{2}}$, then $\alpha\beta = \sqrt{2}$