Problem 1. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by “rotation by 90 degrees counterclockwise”. Is $T$ diagonalizable over $\mathbb{R}$? (Prove your answer to be correct.)

Solution: The linear map $T$ cannot be diagonalizable, since it has no nonzero eigenvectors. Indeed, if $\vec{v}$ were a nonzero eigenvector, then $\vec{v}$ and $T\vec{v}$ would span the same line, and this is not true for any $\vec{v} \in \mathbb{R}^2$.

Problem 2. Fix an integer $n \geq 0$. Let $V = \mathbb{C}[x]_{\leq n} \subset \mathbb{C}[x]$ be the subspace of polynomials of degree $\leq n$. For each $\lambda \in \mathbb{C}$, consider the $\mathbb{C}$-linear operator $T_\lambda : V \to V$ determined by

$$T_\lambda(f(x)) = \frac{d}{dx}(f(x)) - \lambda f(x).$$

Calculate the rank of $T_\lambda : \mathbb{C}[x]_{\leq n} \to \mathbb{C}[x]_{\leq n}$ as a function of $\lambda$ and $n$.

Solution: The answer is that the rank is $n+1$ for $\lambda \neq 0$ and the rank is $n$ for $\lambda = 0$. To see this, we compute the action of $T_\lambda$ in the basis $\{1, x, x^2, \ldots, x^n\}$. We have $T_\lambda(x^m) = -\lambda x^m + mx^{m-1}$, so the matrix of $T_\lambda$ in this basis is

$$
\begin{bmatrix}
-\lambda & & & & \\
& 2 & & & \\
& & -\lambda & & \\
& & & \ddots & \\
& & & & n-1 & -\lambda \\
& & & & & \lambda
\end{bmatrix}.
$$

If $\lambda$ is nonzero, this is a lower triangular matrix with nonzero diagonal, hence invertible and of rank $n+1$. If $\lambda = 0$, then the first $n$ columns are clearly linearly independent and the last column is 0, so the matrix has rank $n$.

Problem 3. Let $R$ be a PID (principal ideal domain). Let $x$ and $y$ in $R$. Let $d$ be a GCD of $x$ and $y$ (meaning that every common divisor of $x$ and $y$ divides $d$) and let $m$ be an LCM of $x$ and $y$ (meaning that every common multiple of $x$ and $y$ is divisible by $m$). Show that $R/xR \oplus R/yR$ is isomorphic (as an $R$-module) to $R/dR \oplus R/mR$.

Solution: We first cover the case that $x$ and $y$ are nonzero. Factor $x$ as $\prod p_i^{e_i}$ and factor $y$ as $\prod p_i^{f_i}$. Then $g = \prod p_i^{\min(e_i, f_i)}$ and $m = \prod p_i^{\max(e_i, f_i)}$. By the Chinese Remainder Theorem

$$
R/xR \cong \bigoplus_i R/p_i^{e_i}R,
R/yR \cong \bigoplus_i R/p_i^{f_i}R,
R/gR \cong \bigoplus_i R/p_i^{\min(e_i, f_i)}R,
R/mR \cong \bigoplus_i R/p_i^{\max(e_i, f_i)}R.
$$

so

$$
R/xR \oplus R/yR \cong \bigoplus_i \left( R/p_i^{e_i}R \oplus R/p_i^{f_i}R \right),
R/xR \oplus R/yR \cong \bigoplus_i \left( R/p_i^{\min(e_i, f_i)}R \oplus R/p_i^{\max(e_i, f_i)}R \right).
$$

But the unordered pair $(e, f)$ is the same as $(\min(e, f), \max(e, f))$, so the summands match.

Now, if $x = 0$ then $g = y$ and $m = 0$, so the result still holds, and similarly if $y = 0$.

Problem 4. Let $A$ be a ring, let $M$ be an $R$-module and let $E = \text{Hom}_A(M, M)$. Show that $M$ can be written as a nontrivial direct sum if and only if there is an element $e \in E$, other than 0 and $\text{Id}$, with $e^2 = e$. 

Solution: If such an $e$ exists, then we claim that $M = eM \oplus (1 - e)M$. Indeed, the identity $m = em + (1 - e)m$ shows that $M = eM + (1 - e)M$. In order to show that $eM \cap (1 - e)M = \{0\}$, suppose that $em = (1 - e)n$. Then $em = e^2m = e(1 - e)n = 0$, so we have shown that the only solution to $em = (1 - e)n$ is $em = (1 - e)n = 0$.

In the reverse direction, suppose that $M = M_1 \oplus M_2$. Define $e : M \to M$ by $e(m_1, m_2) = (m_1, 0)$. This is clearly a map of $R$-modules, and clearly obeys $e^2 = e$.

Problem 5. Let $M$ be a $3 \times 3$ integer matrix and suppose that $\mathbb{Z}^3 / M\mathbb{Z}^3 \cong \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Let $\wedge^2 M$ be the induced map $\wedge^2 \mathbb{Z}^3 \to \wedge^2 \mathbb{Z}^3$. Compute (with proof) the abelian group $(\wedge^2 \mathbb{Z}^3) / (\wedge^2 M)(\wedge^2 \mathbb{Z}^3)$.

Solution: We factor $M$ in Smith normal form as $UDV$ where $D$ is the diagonal matrix with diagonal entries $(6, 2, 1)$ Then, by functoriality, $\wedge^2(M) = \wedge^2(U) \wedge^2(D) \wedge^2(V)$. Using functoriality again, $\wedge^2(U) \wedge^2(U^{-1}) = \text{Id}$, so $\wedge^2(U)$ is invertible, and similarly for $\wedge^2(V)$, so $\wedge^2(\mathbb{Z}^3) / \wedge^2(M) \cong \wedge^2(M) / \wedge^2(D)$. Now, $\wedge^2(D)$ is the diagonal matrix with diagonal entries $(6 \cdot 2, 6 \cdot 1, 2 \cdot 1) = (12, 6, 2)$. So $(\wedge^2 \mathbb{Z}^3) / (\wedge^2 M)(\wedge^2 \mathbb{Z}^3) \cong \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.