Problem 1. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by "rotation by 90 degrees counter-clockwise". Is T diagonalizable over $\mathbb{R}$? (Prove your answer to be correct.)*

Solution: The linear map T cannot be diagonalizable, since it has no nonzero eigenvectors. Indeed, if $\vec{v}$ were a nonzero eigenvector, then $\vec{v}$ and $T\vec{v}$ would span the same line, and this is not true for any $\vec{v} \in \mathbb{R}^2$.

Problem 2. *Fix an integer $n \geq 0$. Let $V = \mathbb{C}[x]_{\leq n} \subset \mathbb{C}[x]$ be the subspace of polynomials of degree $\leq n$. For each $\lambda \in \mathbb{C}$, consider the $\mathbb{C}$-linear operator $T_\lambda : V \to V$ determined by*

$$T_\lambda(f(x)) = \frac{d}{dx}(f(x)) - \lambda f(x).$$

Calculate the rank of $T_\lambda : \mathbb{C}[x]_{\leq n} \longrightarrow \mathbb{C}[x]_{\leq n}$ as a function of λ and n.

Solution: The answer is that the rank is $n+1$ for $\lambda \neq 0$ and the rank is n for $\lambda = 0$. To see this, we compute the action of T_λ in the basis $\{1, x, x^2, \ldots, x^n\}$. We have $T_\lambda(x^m) = -\lambda x^m + mx^{m-1}$, so the matrix of T_λ in this basis is

$$\begin{bmatrix} -\lambda & & & & & \\ n & -\lambda & & & & \\ & n-1 & -\lambda & & & \\ & & \ddots & \ddots & & \\ & & & 2 & -\lambda & \\ & & & & 1 & -\lambda \end{bmatrix}.$$

If λ is nonzero, this is a lower triangular matrix with nonzero diagonal, hence invertible and of rank $n+1$. If $\lambda = 0$, then the first n columns are clearly linearly independent and the last column is 0, so the matrix has rank n.

Problem 3. *Let R be a PID (principal ideal domain). Let x and y in R. Let d be a GCD of x and y (meaning that every common divisor of x and y divides d) and let m be an LCM of x and y (meaning that every common multiple of x and y is divisible by m). Show that $R/xR \oplus R/yR$ is isomorphic (as an R-module) to $R/dR \oplus R/mR$.*

Solution: We first cover the case that x and y are nonzero. Factor x as $\prod p_i^{e_i}$ and factor y as $\prod p_i^{f_i}$. Then $g = \prod p_i^{\min(e_i,f_i)}$ and $m = \prod p_i^{\max(e_i,f_i)}$. By the Chinese Remainder Theorem

$$\begin{aligned} R/xR &\cong \textstyle\bigoplus_i R/p_i^{e_i}R \\ R/yR &\cong \textstyle\bigoplus_i R/p_i^{f_i}R \\ R/gR &\cong \textstyle\bigoplus_i R/p_i^{\min(e_i,f_i)}R \\ R/mR &\cong \textstyle\bigoplus_i R/p_i^{\max(e_i,f_i)}R. \end{aligned}$$

so

$$\begin{aligned} R/xR \oplus R/yR &\cong \textstyle\bigoplus_i \left(R/p_i^{e_i}R \oplus R/p_i^{f_i}R\right) \\ R/xR \oplus R/yR &\cong \textstyle\bigoplus_i \left(R/p_i^{\min(e_i,f_i)}R \oplus R/p_i^{\max(e_i,f_i)}R\right). \end{aligned}$$

But the unordered pair (e, f) is the same as $(\min(e, f),\ \max(e, f))$, so the summands match.

Now, if $x = 0$ then $g = y$ and $m = 0$, so the result still holds, and similarly if $y = 0$.

Problem 4. *Let A be a ring, let M be an R-module and let $E = \mathrm{Hom}_A(M, M)$. Show that M can be written as a nontrivial direct sum if and only if there is an element $e \in E$, other than 0 and Id, with $e^2 = e$.*

Solution: If such an e exists, then we claim that $M = eM \oplus (1-e)M$. Indeed, the identity $m = em + (1-e)m$ shows that $M = eM + (1-e)M$. In order to show that $eM \cap (1-e)M = \{0\}$, suppose that $em = (1-e)n$. Then $em = e^2 m = e(1-e)n = 0$, so we have shown that the only solution to $em = (1-e)n$ is $em = (1-e)n = 0$.

In the reverse direction, suppose that $M = M_1 \oplus M_2$. Define $e : M \to M$ by $e(m_1, m_2) = (m_1, 0)$. This is clearly a map of R-modules, and clearly obeys $e^2 = e$.

Problem 5. *Let M be a 3×3 integer matrix and suppose that $\mathbb{Z}^3/M\mathbb{Z}^3 \cong \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Let $\bigwedge^2 M$ be the induced map $\bigwedge^2 \mathbb{Z}^3 \longrightarrow \bigwedge^2 \mathbb{Z}^3$. Compute (with proof) the abelian group $(\bigwedge^2 \mathbb{Z}^3)/(\bigwedge^2 M)(\bigwedge^2 \mathbb{Z}^3)$.*

Solution: We factor M in Smith normal form as UDV where D is the diagonal matrix with diagonal entries $(6, 2, 1)$ Then, by functoriality, $\bigwedge^2(M) = \bigwedge^2(U) \bigwedge^2(D) \bigwedge^2(V)$. Using functoriality again, $\bigwedge^2(U) \bigwedge^2(U^{-1}) = \mathrm{Id}$, so $\bigwedge^2(U)$ is invertible, and similarly for $\bigwedge^2(V)$, so $\bigwedge^2(\mathbb{Z}^3)/\bigwedge^2(M) \cong \bigwedge^2(M)/\bigwedge^2(D)$. Now, $\bigwedge^2(D)$ is the diagonal matrix with diagonal entries $(6 \cdot 2,\ 6 \cdot 1,\ 2 \cdot 1) = (12, 6, 2)$. So $(\bigwedge^2 \mathbb{Z}^3)/(\bigwedge^2 M)(\bigwedge^2 \mathbb{Z}^3) \cong \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.