Problem 1. In the ring $\mathbb{Z}/2023\mathbb{Z}$, how many elements obey $x^{17} = 1$? We will helpfully tell you that $2023 = 7 \times 17^2$.

Solution. Let $G = (\mathbb{Z}/2023\mathbb{Z})^\times$ be the unit group. Any element satisfying $x^{17} = 1$ belongs to $G$ (its inverse is $x^{16}$). Thus the problem amounts to determining the number of elements of $G$ with order dividing 17. The order of $G$ is

$$\phi(2023) = (7 - 1) \cdot (17^2 - 17) = 6 \cdot 16 \cdot 17.$$ 

It follows from the structure theorem for finite abelian groups that $G$ is isomorphic to $\mathbb{Z}/17\mathbb{Z} \times H$ where $H$ has order 6. Thus there 17 elements of order dividing 17 in $G$.

Problem 2. Let $R \subset S$ be integral domains and suppose that $R = S \cap \text{Frac}(R)$ (the intersection is taken inside $\text{Frac}(S)$). Let $p$ be an element of $R$ which is prime in $S$ (meaning that $p$ is not 0 or a unit and that, if $p$ divides $xy$, then either $p$ divides $x$ or $p$ divides $y$). Show that $p$ is prime in $R$.

Solution. Suppose $x$ and $y$ are elements of $R$ such that $p$ divides $xy$ in $R$. Then $p$ divides $xy$ in $S$, and thus divides either $x$ or $y$ in $S$; say the former. Thus $x/p$ belongs to $\text{Frac}(R) \cap S$, which we are told is $R$. Hence $p$ divides $x$ in $R$. This shows that $p$ is prime in $R$.

Problem 3. Let $V$ be a finite dimensional complex vector space. A linear operator $T$ on $V$ is called indecomposable if there is no decomposition $V = V_1 \oplus V_2$, with $V_1$ and $V_2$ non-zero, such that $T(V_i) \subset V_i$ for $i = 1, 2$. Suppose that $T$ and $T'$ are indecomposable operators on $V$ with equal trace. Show that there is an invertible linear transformation $g$ of $V$ such that $T = gT'g^{-1}$.

Solution. Let $J_m(\lambda)$ be an $m \times m$ Jordan block with $\lambda$ on the diagonal. By the Jordan normal form theorem, there is a basis for $V$ in which the matrix for $T$ has the form

$$
\begin{pmatrix}
J_{m_1}(\lambda_1) & & \\
& \ddots & \\
& & J_{m_r}(\lambda_r)
\end{pmatrix}
$$

We claim that $r = 1$, i.e., there is only one Jordan block. Indeed, suppose $r > 1$. Let $V_1$ be the span of the basis vectors in the first Jordan block, and let $V_2$ be the span of the basis vectors in the remaining blocks. Then each $V_i$ is non-zero and $T(V_i) \subset V_i$, contradicting $T$ being indecomposable. This proves the claim.

We thus see that the matrix for $T$ is a single Jordan block $J_n(\lambda)$, where $n = \dim(V)$. Similarly, the matrix for $T'$ in an appropriate basis is $J_n(\mu)$. Since $T$ and $T'$ have equal traces, we have $\lambda = \mu$. Thus there are bases in which $T$ and $T'$ have the same matrix, which proves the existence of the element $g$.

Problem 4. Let $A$ be an invertible real symmetric matrix. Suppose there is a real number $C$ such that $|\text{Tr}(A^n)| \leq C$ for all integers $n$. Show that $A^2$ is the identity matrix.

Solution. By the spectral theorem, $A$ is diagonalizable with real eigenvalues. Thus, changing our basis if necessary, we may as well assume $A$ is diagonal. Let $\lambda_1, \ldots, \lambda_r$ be its diagonal
entries; these are non-zero since $A$ is invertible. We are given

$$|\text{Tr}(A^n)| = |\lambda_1^n + \cdots + \lambda_r^n| \leq C$$

for all integers $n$. This implies $\lambda_i = \pm 1$ for all $i$. Indeed, if $|\lambda_i| > 1$ for some $i$ then $|\text{Tr}(A^n)|$ would be unbounded as $n$ varies over positive even integers (taking even integers ensures that each $\lambda_i^n$ is positive, and so there is no cancellation). Similarly, if $|\lambda_i| < 1$ then we would find unbounded growth when $n$ is negative and even. We thus see that $A$ is diagonal with diagonal entries $\pm 1$, and so $A^2$ is the identity.

**Problem 5.** Let $V$ be a complex vector space of finite dimension $n$, and let $T: V \to V$ be a diagonalizable linear operator of rank $r$. What is the rank of the operator $\Lambda^k(T): \Lambda^k(V) \to \Lambda^k(V)$? Give a formula for the rank in terms of $n$, $r$, and $k$.

**Solution.** Let $v_1, \ldots, v_n$ be a basis of eigenvectors for $T$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues. Reordering if necessary, we assume that $\lambda_1, \ldots, \lambda_r$ are non-zero and $\lambda_{r+1}, \ldots, \lambda_n$ are zero; note that this $r$ is the rank of $T$, as specified in the problem statement. The space $\Lambda^k(V)$ has a basis consisting of elements $v_{i_1} \wedge \cdots \wedge v_{i_k}$ where $1 \leq i_1 < \cdots < i_k \leq n$. This is in fact an eigenbasis for $\Lambda^k(T)$; the aforementioned basis vector has eigenvalue $\lambda_{i_1} \cdots \lambda_{i_k}$. The rank of $\Lambda^k(T)$ is the number of basis vectors with non-zero eigenvalue. These are exactly the basis vectors with $1 \leq i_1 < \cdots < i_k \leq r$. The number of such vectors is $\binom{n}{k}$, and so this is the rank of $\Lambda^k(T)$. 