Solutions to Algebra 1 QR (May 2023)

Problem 1. In the ring $\mathbb{Z}/2023\mathbb{Z}$, how many elements obey $x^{17} = 1$? We will helpfully tell you that $2023 = 7 \times 17^2$.

Solution. Let $G = (\mathbb{Z}/2023\mathbb{Z})^\times$ be the unit group. Any element satisfying $x^{17} = 1$ belongs to G (its inverse is x^{16}). Thus the problem amounts to determining the number of elements of G with order dividing 17. The order of G is

$$\phi(2023) = (7-1) \cdot (17^2 - 17) = 6 \cdot 16 \cdot 17.$$

It follows from the structure theorem for finite abelian groups that G is isomorphic to $\mathbb{Z}/17\mathbb{Z} \times H$ where H has order $6 \cdot 16$. Thus there 17 elements of order dividing 17 in G.

Problem 2. Let $R \subset S$ be integral domains and suppose that $R = S \cap \operatorname{Frac}(R)$ (the intersection is taken inside $\operatorname{Frac}(S)$). Let p be an element of R which is prime in S (meaning that p is not 0 or a unit and that, if p divides xy, then either p divides x or p divides y). Show that p is prime in R.

Solution. Suppose x and y are elements of R such that p divides xy in R. Then p divides xy in S, and thus divides either x or y in S; say the former. Thus x/p belongs to $\operatorname{Frac}(R) \cap S$, which we are told is R. Hence p divides x in R. This shows that p is prime in R.

Problem 3. Let V be a finite dimensional complex vector space. A linear operator T on V is called *indecomposable* if there is no decomposition $V = V_1 \oplus V_2$, with V_1 and V_2 non-zero, such that $T(V_i) \subset V_i$ for $i = 1, 2$. Suppose that T and T' are indecomposable operators on V with equal trace. Show that there is an invertible linear transformation g of V such that $T = gT'g^{-1}$.

Solution. Let $J_m(\lambda)$ be an $m \times m$ Jordan block with λ on the diagonal. By the Jordan normal form theorem, there is a basis for V in which the matrix for T has the form

$$\begin{pmatrix} J_{m_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{m_r}(\lambda_r) \end{pmatrix}$$

We claim that $r = 1$, i.e., there is only one Jordan block. Indeed, suppose $r > 1$. Let V_1 be the span of the basis vectors in the first Jordan block, and let V_2 be the span of the basis vectors in the remaining blocks. Then each V_i is non-zero and $T(V_i) \subset V_i$, contradicting T being indecomposable. This proves the claim.

We thus see that the matrix for T is a single Jordan block $J_n(\lambda)$, where $n = \dim(V)$. Similarly, the matrix for T' in an appropriate basis is $J_n(\mu)$. Since T and T' have equal traces, we have $\lambda = \mu$. Thus there are bases in which T and T' have the same matrix, which proves the existence of the element g.

Problem 4. Let A be an invertible real symmetric matrix. Suppose there is a real number C such that $|\operatorname{Tr}(A^n)| \le C$ for all integers n. Show that A^2 is the identity matrix.

Solution. By the spectral theorem, A is diagonalizable with real eigenvalues. Thus, changing our basis if necessary, we may as well assume A is diagonal. Let $\lambda_1, \ldots, \lambda_r$ be its diagonal entries; these are non-zero since A is invertible. We are given

$$|\mathrm{Tr}(A^n)| = |\lambda_1^n + \cdots + \lambda_r^n| \leq C$$

for all integers n. This implies $\lambda_i = \pm 1$ for all i. Indeed, if $|\lambda_i| > 1$ for some i then $|\mathrm{Tr}(A^n)|$ would be unbounded as n varies over positive even integers (taking even integers ensures that each λ_i^n is positive, and so there is no cancellation). Similarly, if $|\lambda_i| < 1$ then we would find unbounded growth when n is negative and even. We thus see that A is diagonal with diagonal entries ± 1, and so A^2 is the identity.

Problem 5. Let V be a complex vector space of finite dimension n, and let $T\colon V \to V$ be a diagonalizable linear operator of rank r. What is the rank of the operator $\bigwedge^k(T)\colon \bigwedge^k(V) \to \bigwedge^k(V)$? Give a formula for the rank in terms of n, r, and k.

Solution. Let $v_1, \ldots, v_n$ be a basis of eigenvectors for T. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues. Reordering if necessary, we assume that $\lambda_1, \ldots, \lambda_r$ are non-zero and $\lambda_{r+1}, \ldots, \lambda_n$ are zero; note that this r is the rank of T, as specified in the problem statement. The space $\bigwedge^k(V)$ has a basis consisting of elements $v_{i_1} \wedge \cdots \wedge v_{i_k}$ where $1 \leq i_1 < \cdots < i_k \leq n$. This is in fact an eigenbasis for $\bigwedge^k(T)$; the aforementioned basis vector has eigenvalue $\lambda_{i_1} \cdots \lambda_{i_k}$. The rank of $\bigwedge^k(T)$ is the number of basis vectors with non-zero eigenvalue. These are exactly the basis vectors with $1 \leq i_1 < \cdots < i_k \leq r$. The number of such vectors is $\binom{r}{k}$, and so this is the rank of $\bigwedge^k(T)$.