Algebra 1

Identification Number: ____________

Each problem occurs on a separate page. If possible, please write your solution on that page and its reverse side. More paper, and paper for scrap work, is available.

Problem 1. In the ring $\mathbb{Z}/2023\mathbb{Z}$, how many elements obey $x^{17} = 1$? We will helpfully tell you that $2023 = 7 \times 17^2$.

Problem 2. Let $R \subset S$ be integral domains and suppose that $R = S \cap \mathrm{Frac}(R)$ (the intersection is taken inside $\mathrm{Frac}(S)$). Let p be an element of R which is prime in S (meaning that p is not 0 or a unit and that, if p divides xy, then either p divides x or p divides y). Show that p is prime in R.

Problem 3. Let V be a finite dimensional complex vector space. A linear operator T on V is called *indecomposable* if there is no decomposition $V = V_1 \oplus V_2$, with V_1 and V_2 non-zero, such that $T(V_i) \subset V_i$ for $i = 1, 2$. Suppose that T and T' are indecomposable operators on V with equal trace. Show that there is an invertible linear transformation g of V such that $T = gT'g^{-1}$.

Problem 4. Let A be an invertible real symmetric matrix. Suppose there is a real number C such that $|\mathrm{Tr}(A^n)| \leq C$ for all integers n. Show that A^2 is the identity matrix.

Problem 5. Let V be a complex vector space of finite dimension n, and let $T\colon V \to V$ be a diagonalizable linear operator of rank r. What is the rank of the operator $\bigwedge^k(T)\colon \bigwedge^k(V) \to \bigwedge^k(V)$? Give a formula for the rank in terms of n, r, and k.