Problem 1. Let $p$ be prime and let $\mathbb{F}_p$ be the field with $p$ elements. Let $G$ be the group $\text{GL}_n(\mathbb{F}_p)$ with $n \geq 2$ and let $G$ act on $(\mathbb{F}_p^n)^2$ in the obvious way. How many orbits does $G$ have on $(\mathbb{F}_p^n)^2$? (The “obvious way” is that, for $g \in \text{GL}_n(\mathbb{F}_p)$, and $\vec{x}$ and $\vec{y} \in \mathbb{F}_p^n$, we have $g \ast (\vec{x}, \vec{y}) = (g \vec{x}, g \vec{y})$.)

Problem 2. Let $G$ be a group of order 2023. Show that $G$ is abelian. We will helpfully tell you that $2023 = 7 \times 17^2$.

Problem 3. Let $n$ be a positive integer. The dihedral group of order $2n$, written $D_{2n}$, is defined to be the group generated by two elements $\rho$ and $\sigma$, modulo the relations $\sigma^2 = \rho^n = e$ and $\sigma \rho = \rho^{-1} \sigma$. Show that the abelianization of $D_{2n}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if $n$ is odd and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ if $n$ is even. (The abelianization of a group $G$ is the quotient of $G$ by the subgroup generated by all elements of the form $ghg^{-1}h^{-1}$.)

Problem 4. Let $L/\mathbb{Q}$ be a Galois extension of degree $2^n$ for some positive integer $n$. Show that there is some nonsquare rational number $D$ such that $\sqrt{D} \in L$.

Problem 5. Let $L$ be the field $\mathbb{C}(x_1, x_2, \ldots, x_n)$; in other words, the field of rational functions in $n$ algebraically independent variables $x_1, x_2, \ldots, x_n$ with coefficients in $\mathbb{C}$. Let $K$ be the subfield $\mathbb{C}(x_1^2, x_2^2, \ldots, x_n^2)$. Show that $K(x_1 + x_2 + \cdots + x_n) = L$. (In other words, show that $x_1 + x_2 + \cdots + x_n$ is a primitive element for the extension $L/K$.)