Algebra 2

Problem 1. Let p be prime and let $\mathbb{F}_p$ be the field with p elements. Let G be the group $\mathrm{GL}_n(\mathbb{F}_p)$ with $n \geq 2$ and let G act on $(\mathbb{F}_p^n)^2$ in the obvious way. How many orbits does G have on $(\mathbb{F}_p^n)^2$? (The "obvious way" is that, for $g \in \mathrm{GL}_n(\mathbb{F}_p)$, and $\vec{x}$ and $\vec{y} \in \mathbb{F}_p^n$, we have $g * (\vec{x},\ \vec{y}) = (g\vec{x},\ g\vec{y})$.)

Problem 2. Let G be a group of order 2023. Show that G is abelian. We will helpfully tell you that $2023 = 7 \times 17^2$.

Problem 3. Let n be a positive integer. The dihedral group of order $2n$, written D_{2n}, is defined to be the group generated by two elements ρ and σ, modulo the relations $\sigma^2 = \rho^n = e$ and $\sigma\rho = \rho^{-1}\sigma$. Show that the abelianization of D_{2n} is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if n is odd and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ if n is even. (The abelianization of a group G is the quotient of G by the subgroup generated by all elements of the form $ghg^{-1}h^{-1}$.)

Problem 4. Let $L/\mathbb{Q}$ be a Galois extension of degree 2^n for some positive integer n. Show that there is some nonsquare rational number D such that $\sqrt{D} \in L$.

Problem 5. Let L be the field $\mathbb{C}(x_1, x_2, \ldots, x_n)$; in other words, the field of rational functions in n algebraically independent variables x_1, x_2, $\ldots$, x_n with coefficients in $\mathbb{C}$. Let K be the subfield $\mathbb{C}(x_1^2, x_2^2, \ldots, x_n^2)$. Show that $K(x_1 + x_2 + \cdots x_n) = L$. (In other words, show that $x_1 + x_2 + \cdots + x_n$ is a primitive element for the extension L/K.)