Algebra 1 Solutions

Problem. Let a and b be relatively prime positive integers. Let R be a ring in which $ab = 0$. Show that $R \cong A \times B$, where A is a ring in which $a = 0$ and B is a ring in which $b = 0$. (The rings R, A and B need not be commutative.)

Solution. By the Chinese remainder theorem, we have

$$\mathbb{Z}/ab\mathbb{Z} = \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}.$$

Let $c, d \in \mathbb{Z}/ab\mathbb{Z}$ correspond to $(1,0)$ and $(0,1)$ under this isomorphism. Then c and d are central idempotent elements of R that multiply to zero and sum to 1. Letting $A = cR$ and $B = dR$, we have $R = A \times B$. Since $ac = 0$ (in $\mathbb{Z}/ab\mathbb{Z}$, and thus in R) and c is the identity in A, we have $a = 0$ in A; similarly, $b = 0$ in B.

Problem. Let M be an $n \times n$ matrix with entries in $\mathbb{Q}$, such that $M^3 = 2\,\mathrm{Id}_n$. Show that n is divisible by 3.

Solution. Let $a = \det(M) \in \mathbb{Q}$. From the given equation, we find $a^3 = 2^n$. Looking at the prime factorization of a, it follows that n is divisible by 3.

Second solution. Let $F = \mathbb{Q}[M]$ be the subring of the matrix ring $M_n(\mathbb{Q})$ generated by $\mathbb{Q}$ and M. This is isomorphic to $\mathbb{Q}[x]/(x^3 - 2)$, which is a field of degree 3 over $\mathbb{Q}$, since $x^3 - 2$ is irreducible. Since $\mathbb{Q}^n$ is a module over F, i.e., an F-vector space, it is isomorphic to F^m for some m. Since F is 3-dimensional over $\mathbb{Q}$, this gives $n = 3m$.

Problem. Let k be a field and let V be a finite dimensional vector space over k. Let u, v, x and y be nonzero vectors in V such that $u \otimes v = x \otimes y$ in the tensor product $V \otimes_k V$. Show that there is a nonzero scalar c in k such that $x = cu$ and $y = c^{-1}v$.

Solution. Let $u_1, \dots, u_n$ be a basis for V with $u_1 = u$ and $x \in \mathrm{span}(u_1, u_2)$. Let $v_1, \dots, v_n$ be a second basis for V with $v_1 = v$ and $y \in \mathrm{span}(v_1, v_2)$. Then $u_i \otimes v_j$ is a basis for $V \otimes V$. Write $x = au_1 + bu_2$ and $y = cv_1 + dv_2$. Then the given equation becomes

$$u_1 \otimes v_1 = (au_1 + bu_2) \otimes (cv_1 + dv_2).$$

Equating coefficients of $u_i \otimes v_j$, we find $b = d = 0$ and $ac = 1$, which completes the proof.

Problem. Let A and B be principal ideal domains, with $A \subset B$. Let x and y be elements of A which are relatively prime in A. Show that x and y are relatively prime in B. (We say that two elements of an integral domain R are relatively prime in R if the only elements of R which divide both of them are units.)

Solution. In general, elements p and q in a PID R are relatively prime if and only if they satisfy the Bézout identity, i.e., there exist $a, b \in R$ such that $ap + bq = 1$. Applying this with $R = A$ and $p = x$ and $q = y$, we see that there are $a, b \in A$ such that $ax + by = 1$. This equation continues to hold in B, and so x and y are relatively prime there too.

Problem. Let R be an integral domain, and let S be a subring of R. Show that the following are equivalent:

(1) The natural inclusion $\mathrm{Frac}(S) \hookrightarrow \mathrm{Frac}(R)$ is an isomorphism. (Here $\mathrm{Frac}(R)$ is the field of fractions of R, and likewise for $\mathrm{Frac}(S)$.)
(2) The S-module R/S is torsion, meaning that, for every $x \in R/S$, there is a nonzero $s \in S$ with $sx = 0$.

Solution. Suppose (1) holds. Let x be an element of R. By assumption, $x = a/s$ for some $a, s \in S$ with $s \neq 0$, which means $sx = a \in S$. Thus $sx = 0$ in R/S.

Now suppose (2) holds. Let x be an element of R. By assumption, there exists $s \in S$ non-zero such that $sx = 0$ in R/S, meaning $sx = a$ holds in R for some $a \in S$. We thus see that $x = a/s$ holds in $\mathrm{Frac}(R)$. If y is a second element of R that is non-zero, we can similarly write $y = a'/s'$, and so $x/y = (as')/(sa')$ belongs to $\mathrm{Frac}(S)$.