Algebra 1

Problem 1. Let a and b be relatively prime positive integers. Let R be a ring in which $ab = 0$. Show that $R \cong A \times B$, where A is a ring in which $a = 0$ and B is a ring in which $b = 0$. (The rings R, A and B need not be commutative.)

Problem 2. Let M be an $n \times n$ matrix with entries in $\mathbb{Q}$, such that $M^3 = 2\,\mathrm{Id}_n$. Show that n is divisible by 3.

Problem 3. Let k be a field and let V be a vector space over k. Let u, v, x and y be nonzero vectors in V such that $u \otimes v = x \otimes y$ in the tensor product $V \otimes_k V$. Show that there is a nonzero scalar c in k such that $x = cu$ and $y = c^{-1}v$.

Problem 4. Let A and B be principal ideal domains, with $A \subset B$. Let x and y be elements of A which are relatively prime in A. Show that x and y are relatively prime in B. (We say that two elements of an integral domain R are relatively prime in R if the only elements of R which divide both of them are units.)

Problem 5. Let R be an integral domain, and let S be a subring of R. Show that the following are equivalent:

(1) The natural inclusion $\mathrm{Frac}(S) \hookrightarrow \mathrm{Frac}(R)$ is an equality. (Here $\mathrm{Frac}(R)$ is the field of fractions of R, and likewise for $\mathrm{Frac}(S)$.)

(2) The S-module R/S is torsion, meaning that, for every $x \in R/S$, there is a nonzero $s \in S$ with $sx = 0$.