Algebra II Exam: May 2022

Please wait until directed to begin the exam. Please use the indicated page, and its reverse side, for your solution to each problem. Extra pages are attached at the end if you need more space; please indicate if you have used them.

Please write your **identification number** here:

Have fun!

Problem 1. Let G be a simple group. Let H be a normal subgroup of $G \times G$. Show that H is isomorphic to either the trivial group, to G or to $G \times G$.

Problem 2. Let p be a prime. Let G be a group such that $|G|$ is divisible by p but not by p^2. Show that G contains at most $p-1$ conjugacy classes of elements of order p.

Problem 3. Let p be a prime. Let G be a subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ whose order is prime to p. Let $\pi : \mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ be the reduction modulo p map. Show that there is a group homomorphism $\sigma : G \to \mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$ such that $\pi(\sigma(g)) = g$ for all $g \in G$.

Problem 4. Let ζ be a primitive 25th root of 1 over $\mathbb{Q}$. Show that the equation $X^5 - 5$ has no solutions over $\mathbb{Q}[\zeta]$.

Problem 5. Let p be a prime, let k be a field in which $p \neq 0$ and let $f(x)$ be the polynomial $\frac{x^p-1}{x-1} = x^{p-1} + x^{p-2} + \cdots + x^2 + x + 1$. Let $g_1(x)g_2(x)\cdots g_r(x)$ be the factorization of $f(x)$ into irreducibles in $k[x]$. Show that all the polynomials $g_i(x)$ have the same degree.