Problem 1. Let $A$ be an $n \times n$ integer matrix and let $A^T$ be its transpose. Let $X$ and $Y$ be the abelian groups $X = \mathbb{Z}^n/\Lambda \mathbb{Z}^n$ and $Y = \mathbb{Z}^n/A^T \mathbb{Z}^n$. Show that $X$ and $Y$ are isomorphic as abelian groups.

Solution: Write $A$ in Smith normal form as $A = UDV$ where

$$D = \begin{bmatrix} d_1 & & \\
& d_2 & \\
& & \ddots \\
& & & d_n \end{bmatrix}$$

with $U$ and $V$ invertible. So $A^T = V^T D^T U^T = V^T D U^T$ since $D$ is diagonal. Then $X \cong \mathbb{Z}^n/D\mathbb{Z}^n \cong \mathbb{Z}^n/D^T\mathbb{Z}^n \cong Y$.

Problem 2. Let $k$ be a field. For each of the following rings, determine if it is a PID or a UFD or neither or both.

1. $k[x, y]$.
2. $k[x, y]/(xy - 1)k[x, y]$.
3. $k[x, y]/(y^2 - x^3)k[x, y]$.

Solution

1. $k[x, y]$ This ring is a UFD, since $k[x]$ is a UFD and, if $A$ is a UFD, then $A[y]$ is as well. It is not a PID, since the ideal $(x, y)$ is not principal.

2. $k[x, y]/(xy - 1)$ This ring is a PID and hence a UFD. Note that this ring is isomorphic to the Laurent polynomial ring $k[x, x^{-1}]$. Let $I$ be a nonzero ideal of $k[x, x^{-1}]$ and let $J = k[x] \cap I$. Since $k[x]$ is a PID, we have $J = f(x)k[x]$ for some polynomial $f$, and therefore $f(x)k[x, x^{-1}] \subseteq I$. We claim that, in fact, $f(x)k[x, x^{-1}] \subseteq I$. To see this, let $g(x) \in I$. Then there is some positive integer $N$ such that $x^N g(x) \in k[x]$, so $f(x)$ divides $x^N g(x)$ in $k[x]$. Then $f(x)$ also divides $g(x)$ in $k[x, x^{-1}]$. So we have shown that $f(x)k[x, x^{-1}] \subseteq I$ and $I$ is principal.

3. This ring is not a UFD, and therefore not a PID. We claim that $x$ and $y$ are non-associate irreducibles, so the equation $y^2 = x^3$ is a non-unique factorization. To see that $x$ and $y$ are irreducible, note that this ring is isomorphic to the subring $k[t^2, t^3]$ of $k[t]$.

Problem 3. Let $T$ be an $(n \times n)$-matrix over an algebraically closed field $k$ of characteristic $p$. Assume that all eigenvalues of $T$ lie in $\mathbb{F}_p \subset k$. Is the matrix $T^p - T$ nilpotent? If yes, give a proof; if not, give an example.

Solution: Yes, $T^p - T$ is nilpotent. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the generalized eigenvalues of $T$. Note that each generalized eigenvalue of $T$ is an eigenvalue (possibly with different multiplicity), so each $\lambda_j$ lies in $\mathbb{F}_p$, so $\lambda_j^p - \lambda_j = 0$ for each $j$. Then the generalized eigenvalues of $T^p - T$ are all $\lambda_j^p - \lambda_j = 0$. So the characteristic polynomial of $T^p - T$ is $x^p$ and so $T^p - T$ is nilpotent.

Problem 4. Calculate the number of subgroups $L \subset \mathbb{Z}^3$ with $\mathbb{Z}^3/L$ being isomorphic abstractly to $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

Solution: The answer is $5^2 + 5 + 1 = 31$. There are many ways to derive this; here is one. If $\mathbb{Z}^3/L \cong (\mathbb{Z}/5\mathbb{Z})^3$, then $L \supset (5\mathbb{Z})^3$. So $L$ is determined by its image in the quotient $\mathbb{Z}^3/(5\mathbb{Z})^3 \cong (\mathbb{Z}/5\mathbb{Z})^3$. The image of $L$ in $(\mathbb{Z}/5\mathbb{Z})^3$ must be a one dimensional subspace of the vector space $\mathbb{F}_5^3$. Such a subspace is generated by some $(x, y, z) \in \mathbb{F}_5^3$ with $x$, $y$ and $z$ not all
0. The number of ways to choose \((x, y, z)\) is \(5^3 - 1\), but rescaling this vector gives the same subspace of \(\mathbb{F}_5^3\), so \(\frac{5^3 - 1}{5 - 1} = 31\).

**Problem 5.** Let \(R\) be a PID which is free as rank \(n\) as a \(\mathbb{Z}\)-module and let \(\pi\) be a prime element of \(R\). Show that \(|R/\pi R|\) is of the form \(p^k\) for some prime integer \(p\) and some \(1 \leq k \leq n\). (Remark: Note that units, and the zero element, are not considered prime.)

**Solution:** We first check that \(\pi R \cap \mathbb{Z}\) is not \((0)\). Indeed, since \(R\) is of finite rank as a \(\mathbb{Z}\)-module, we must have a relation \(\pi^n = a_{n-1}\pi^{n-1} + \cdots + a_1\pi + a_0\) for \(a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z}\). Since \(\pi \neq 0\) and \(R\) is a domain, we can assume that \(a_0 \neq 0\). Then \(a_0 \in \pi R\). So \(\pi R \cap \mathbb{Z}\) is not the zero ideal.

Since \(\mathbb{Z}\) is a PID, \(\pi R \cap \mathbb{Z}\) must be \(g\mathbb{Z}\) for some \(g \in \mathbb{Z}\). We claim that \(g\) is prime. If not, let \(g = ab\) be a nontrivial factorization, then \(ab = 0\) in \(R/\pi R\) and neither \(a\) nor \(b\) is \(0\) in \(R/\pi R\), a contradiction. So \(R \cap \mathbb{Z} = p\mathbb{Z}\) for some prime \(p\). Then \(R/\pi R\) is an \(\mathbb{F}_p\) vector space, so \(|R/\pi R| = p^k\) for some \(k\).