Problem 1. Let A be an $n \times n$ integer matrix and let A^T be its transpose. Let X and Y be the abelian groups $X = \mathbb{Z}^n / A\mathbb{Z}^n$ and $Y = \mathbb{Z}^n / A^T \mathbb{Z}^n$. Show that X and Y are isomorphic as abelian groups.

Solution: Write A in Smith normal form as $A = UDV$ where

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

with U and V invertible. So $A^T = V^T D^T U^T = V^T D U^T$ since D is diagonal. Then $X \cong \mathbb{Z}^n / D\mathbb{Z}^n \cong \mathbb{Z}^n / D^T \mathbb{Z}^n \cong Y$.

Problem 2. Let k be a field. For each of the following rings, determine if it is a PID or a UFD or neither or both.

(1) $k[x, y]$.
(2) $k[x, y]/(xy - 1)k[x, y]$.
(3) $k[x, y]/(y^2 - x^3)k[x, y]$.

Solution

(1) $k[x, y]$ This ring is a UFD, since $k[x]$ is a UFD and, if A is a UFD, then $A[y]$ is as well. It is not a PID, since the ideal $\langle x, y \rangle$ is not principle.
(2) $k[x, y]/(xy-1)$ This ring is a PID and hence a UFD. Note that this ring is isomorphic to the Laurent polynomial ring $k[x, x^{-1}]$. Let I be a nonzero ideal of $k[x, x^{-1}]$ and let $J = k[x] \cap I$. Since $k[x]$ is a PID, we have $J = f(x)k[x]$ for some polynomial f, and therefore $f(x)k[x, x^{-1}] \subseteq I$. We claim that, in fact, $f(x)k[x, x^{-1}] \subseteq I$. To see this, let $g(x) \in I$. Then there is some positive integer N such that $x^N g(x) \in k[x]$, so $f(x)$ divides $x^N g(x)$ in $k[x]$. Then $f(x)$ also divides $g(x)$ in $k[x, x^{-1}]$. So we have shown that $f(x)k[x, x^{-1}] \subseteq I$ and I is principal.
(3) This ring is not a UFD, and therefore not a PID. We claim that x and y are non-associate irreducibles, so the equation $y^2 = x^3$ is a non-unique factorization. To see that x and y are irreducible, note that this ring is isomorphic to the subring $k[t^2, t^3]$ of $k[t]$.

Problem 3. Let T be an $(n \times n)$-matrix over an algebraically closed field k of characteristic p. Assume that all eigenvalues of T lie in $\mathbb{F}_p \subset k$. Is the matrix $T^p - T$ nilpotent? If yes, give a proof; if not, give an example.

Solution: Yes, $T^p - T$ is nilpotent. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the generalized eigenvalues of T. Note that each generalized eigenvalue of T is an eigenvalue (possibly with different multiplicity), so each λ_j lies in $\mathbb{F}_p$, so $\lambda_j^p - \lambda_j = 0$ for each j. Then the generalized eigenvalues of $T^p - T$ are all $\lambda_j^p - \lambda_j = 0$. So the characteristic polynomial of $T^p - T$ is x^p and so $T^p - T$ is nilpotent.

Problem 4. Calculate the number of subgroups $L \subset \mathbb{Z}^3$ with $\mathbb{Z}^3/L$ being isomorphic abstractly to $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

Solution: The answer is $5^2 + 5 + 1 = 31$. There are many ways to derive this; here is one.

If $\mathbb{Z}^3/L \cong (\mathbb{Z}/5\mathbb{Z})^2$, then $L \supset (5\mathbb{Z})^3$. So L is determined by its image in the quotient $\mathbb{Z}^3/(5\mathbb{Z})^3 \cong (\mathbb{Z}/5\mathbb{Z})^3$. The image of L in $(\mathbb{Z}/5\mathbb{Z})^3$ must be a one dimensional subspace of the vector space $\mathbb{F}_5^3$. Such a subspace is generated by some $(x, y, z) \in \mathbb{F}_5^3$ with x, y and z not all

0. The number of ways to choose (x, y, z) is $5^3 - 1$, but resclaing this vector gives the same subspace of $\mathbb{F}_5^3$, so $\frac{5^3-1}{5-1} = 31$.

Problem 5. Let R be a PID which is free as rank n as a $\mathbb{Z}$-module and let π be a prime element of R. Show that $|R/\pi R|$ is of the form p^k for some prime integer p and some $1 \leq k \leq n$. (Remark: Note that units, and the zero element, are not considered prime.)

Solution: We first check that $\pi R \cap \mathbb{Z}$ is not (0). Indeed, since R is of finite rank as a $\mathbb{Z}$-module, we must have a relation $\pi^n = a_{n-1}\pi^{n-1} + \cdots + a_1\pi + a_0$ for a_0, a_1, $\ldots$, $a_{n-1} \in \mathbb{Z}$. Since $\pi \neq 0$ and R is a domain, we can assume that $a_0 \neq 0$. Then $a_0 \in \pi R$. So $\pi R \cap \mathbb{Z}$ is not the zero ideal.

Since $\mathbb{Z}$ is a PID, $\pi R \cap \mathbb{Z}$ must be $g\mathbb{Z}$ for some $g \in \mathbb{Z}$. We claim that g is prime. If not, let $g = ab$ be a nontrivial factorization, then ab is 0 in $R/\pi R$ and neither a nor b is 0 in $R/\pi R$, a contradiction. So $R \cap \mathbb{Z} = p\mathbb{Z}$ for some prime p. Then $R/\pi R$ is an $\mathbb{F}_p$ vector space, so $|R/\pi R| = p^k$ for some k.