ALGEBRA I

Problem 1. Let V be a finite dimensional vector space and let $A : V \to V$ be a linear map. Show that $\dim \operatorname{Ker}(A^2) \leq 2 \dim \operatorname{Ker}(A)$.

Problem 2. Let A and B be 3×3 complex matrices, and suppose that 1 is the only eigenvalue of A. For a non-negative integer n, let $f(n) = \operatorname{Tr}(BA^n)$. Show that f is a polynomial function of n.

Problem 3. Let R be a unique factorization domain (UFD) and let u and v be two nonzero elements of R.

(1) Prove or disprove: The ideal $uR \cap vR$ is necessarily principal.

(2) Prove or disprove: The ideal $uR + vR$ is necessarily principal.

Problem 4. Let S be a principal ideal domain (PID) and let a, b and c be nonzero elements of S. Show that $aS \cap (bS + cS) = (aS \cap bS) + (aS \cap cS)$.

Problem 5. Let M and N be finitely generated $\mathbb{Z}$ modules and let $h : M \to N$ be a $\mathbb{Z}$-linear homorphism. For a prime number p, let

$$h_p : M \otimes_{\mathbb{Z}} \mathbb{F}_p \to N \otimes_{\mathbb{Z}} \mathbb{F}_p$$

be the map $h \otimes \operatorname{Id}$. We consider h_p as a map of $\mathbb{F}_p$-vector spaces. Show that there is an integer d (depending on M, N and h) such that $\dim \operatorname{Ker} h_p = d$ for all sufficiently large p.

Date: Aug 2022.