Algebra 1

Problem 1. Let $V$ be a 2-dimensional complex vector space. What is the largest value of $n$ for which there are vectors $v_1, \ldots, v_n$ in $V$ such that $v_1 \otimes^3 \ldots \otimes^3 v_n$ are linearly independent? Here $v \otimes^3$ denotes the element $v \otimes v \otimes v$ of $V \otimes^3 = V \otimes V \otimes V$.

Solution. Let $e, f$ be a basis for $V$. Consider an element $v = \alpha e + \beta f$ of $V$. Then
\[ v \otimes^3 = \alpha^3 eee + \alpha^2 \beta (efe + efe + fee) + \alpha \beta^2 (eff + ffe + ffe) + \beta^3 fff. \]
Here we have omitted tensor symbols; thus $eee$ means $e \otimes e \otimes e$. In other words, if we define
\[ g_1 = eee, \quad g_2 = eef + efe + ffe, \quad g_3 = eff + ffe + ffe, \quad g_4 = fff \]
then
\[ v \otimes^3 = \alpha^3 g_1 + \alpha^2 \beta g_2 + \alpha \beta^2 g_3 + \beta^3 g_4. \]
We thus see that $v \otimes^3$ belongs to the span of $g_1, \ldots, g_4$, which is a four dimensional space (note that the $g$'s are linearly independent since they have no basis vectors in common). This shows that $n \leq 4$.

In fact, $n = 4$. To see this, let $v_1, \ldots, v_4$ be four elements of $V$ and write $v_i = \alpha_i e + \beta_i f$.
Expressing $v_i \otimes^3$ in terms of the $g$ basis, the coefficient vectors are the rows of the following matrix:
\[
\begin{pmatrix}
\alpha_1^3 & \alpha_1^2 \beta & \alpha_1 \beta^2 & \beta^3 \\
\alpha_2^3 & \alpha_2^2 \beta & \alpha_2 \beta^2 & \beta^2 \\
\alpha_3^3 & \alpha_3^2 \beta & \alpha_3 \beta^2 & \beta^3 \\
\alpha_4^3 & \alpha_4^2 \beta & \alpha_4 \beta^2 & \beta^3
\end{pmatrix}
\]
The $v_i \otimes^3$ are linearly independent if and only if the above matrix is non-singular. We thus just need to pick the $\alpha$'s and $\beta$'s to make the determinant non-zero. This is clearly possible, since the determinant is not the zero polynomial: the coefficient of $\alpha_1^3 \alpha_2^2 \alpha_3$ is non-zero (it appears in only one term when we expand the determinant). To be definite, we can take
\[ (\alpha_1, \beta_1) = (1, 0), \quad (\alpha_2, \beta_2) = (1, 1), \quad (\alpha_3, \beta_3) = (1, -1), \quad (\alpha_4, \beta_4) = (0, 1). \]

Remark. For any complex vector space $V$, the vectors $v \otimes^d$ belong to and span the space $\text{Sym}^d(V)$, which we identify with the $S_d$-invariant vectors of $V \otimes^d$. Thus the maximal $n$ for which there exists linearly independent vectors $v_1 \otimes^3, \ldots, v_n \otimes^3$ is given by $n = \dim \text{Sym}^d(V)$. Explicitly, this is $\binom{m+d-1}{d}$ where $m = \dim(V)$.

Problem 2. Let $X$ be an $n \times n$ matrix with entries in $\mathbb{C}$. Let
\[ V = \{ Y \in \text{Mat}_{n \times n}(\mathbb{C}) \mid XY = YX \}, \]
which is a vector subspace of $\text{Mat}_{n \times n}(\mathbb{C})$. Show that $\dim_{\mathbb{C}} V \geq n$.

Solution 1. Regard $\mathbb{C}^n$ as a $\mathbb{C}[t]$-module with $t$ acting by $X$. Then $V$ is exactly the set of $\mathbb{C}[t]$-module endomorphisms of $\mathbb{C}^n$. Thus it suffices to prove the following statement: if $M$ is a finite dimensional $\mathbb{C}[t]$-module then $\dim \text{End}_{\mathbb{C}[t]}(M) \geq \dim M$. (Throughout this solution, “dimension” means “dimension as a $\mathbb{C}$-vector space.”)

Suppose that $M$ and $N$ are finite dimensional $\mathbb{C}[t]$-modules. Then $\text{End}_{\mathbb{C}[t]}(M \oplus N)$ contains $\text{End}_{\mathbb{C}[t]}(M) \oplus \text{End}_{\mathbb{C}[t]}(N)$. Thus if the result is true for $M$ and $N$ then it is true for $M \oplus N$.  

1
By the structure theorem, every finite dimensional \( \mathbb{C}[t] \)-module is a direct sum of finite dimensional cyclic \( \mathbb{C}[t] \)-modules. It thus suffices to prove the result for such modules. Now, if \( R \) is any commutative ring and \( I \) is an ideal then \( \text{End}_R(R/I) = R/I \). In particular, for \( M = \mathbb{C}[t]/I \), with \( I \) a non-zero ideal, we see that \( \text{End}_{\mathbb{C}[t]}(M) \cong M \), and so the result holds.

**Solution 2.** Write \( V(X) \) for the space \( V \) in the problem. Suppose that \( X \) is a block matrix

\[
X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}
\]

where \( A \) has size \( a \times a \) and \( B \) has size \( b \times b \), with \( a+b = n \). Then \( V(X) \) contains \( V(A) \oplus V(B) \).

(Here we think of elements of \( V \) as column vectors. Define the \( \mathbb{C} \)-module \( \mathbb{C}[t] \). We write \( = \mathbb{M} \) dimension cyclic \( \mathbb{C} \). If \( R/V \) (Here we think of elements of \( A \) has exactly 168 elements of order 10. What is the order of \( \mathbb{Z} \) in the problem? Suppose that \( x = (y, z) \) of \( A \). Then \( x \) has order 10 if and only if \( y \) and \( z \) are both non-zero. We thus see that the number of elements of \( A \) of order \( 10 \) is \((2^m - 1)(5^m - 1)\). We therefore have \((2^m - 1)(5^m - 1) = 168\). The only solution to this equation is \((n, m) = (3, 2)\). (Reason: For \( m \geq 3 \), the number \( 5^m - 1 \) does not divide 168, so \( m \) must be \( 1 \) or \( 2 \). Since \( 168/(5^1 - 1) = 42 \) is not of the form \( 2^n - 1 \), we cannot have \( m = 1 \).) Hence \( A = (\mathbb{Z}/2)^3 \times (\mathbb{Z}/5)^2 \) has order \( 2^3 \times 5^2 = 200 \).

**Problem 4.** Let \( A \) be a finite abelian group such that \( a^{10} = 1 \) for all \( a \) in \( A \). Suppose that \( A \) has exactly 168 elements of order 10. What is the order of \( A \)?

**Solution.** We write \( A \) additively; thus we have \( 10x = 0 \) for all \( x \in A \). By the structure theorem, we have \( A \cong \mathbb{Z}/p_1^{e_1} \times \cdots \times \mathbb{Z}/p_r^{e_r} \) for prime numbers \( p_1, \ldots, p_r \) and positive integers \( e_1, \ldots, e_r \). Since \( 10x = 0 \) for all \( x \in A \), we must have \( p_i^{e_i} \mid 10 \) for all \( i \), and so \( p_i \in \{2, 5\} \) and \( e_i = 1 \). We thus have \( A = (\mathbb{Z}/2)^n \times (\mathbb{Z}/5)^m \) for non-negative integers \( n \) and \( m \). Consider an element \( x = (y, z) \) of \( A \). Then \( x \) has order 10 if and only if \( y \) and \( z \) are both non-zero. We thus see that the number of elements of \( A \) of order 10 is \((2^n - 1)(5^m - 1)\). We therefore have \((2^n - 1)(5^m - 1) = 168\). The only solution to this equation is \((n, m) = (3, 2)\). (Reason: For \( m \geq 3 \), the number \( 5^m - 1 \) does not divide 168, so \( m \) must be \( 1 \) or \( 2 \). Since \( 168/(5^1 - 1) = 42 \) is not of the form \( 2^n - 1 \), we cannot have \( m = 1 \).) Hence \( A = (\mathbb{Z}/2)^3 \times (\mathbb{Z}/5)^2 \) has order \( 2^3 \times 5^2 = 200 \).

**Problem 5.** Let \( S = \mathbb{Q}[t] \). We’ll write elements of \( S^\otimes 2 \) as column vectors. Define the
following \(S\)-modules:

\[
M_1 = S^{\oplus 2}/(S[\begin{array}{c}1 \\ 1 \end{array}] + S[\begin{array}{c}0 \\ 1 \end{array}]) \\
M_2 = S^{\oplus 2}/(S[\begin{array}{c}1 \\ 0 \end{array}] + S[\begin{array}{c}0 \\ 0 \end{array}]) \\
M_3 = S^{\oplus 2}/(S[\begin{array}{c}1 \\ -1 \end{array}] + S[\begin{array}{c}0 \\ 1 \end{array}]) \\
M_4 = S^{\oplus 2}/(S[\begin{array}{c}1 \\ -1 \end{array}] + S[\begin{array}{c}0 \\ 0 \end{array}])
\]

Two of these modules are isomorphic to each other. Prove that they are isomorphic, and show that the other pairs of modules are nonisomorphic.

**Solution.** We decompose each of the modules according to the structure theorem. Clearly, we have

\[
M_1 = S/tS \oplus S/tS, \quad M_2 = S/tS \oplus S/(t-1)S.
\]

We now consider \(M_3\). Let \(e_1 = [\begin{array}{c}1 \\ 0 \end{array}]\) and \(e_2 = [\begin{array}{c}0 \\ 1 \end{array}]\) be the standard basis of \(S^{\oplus 2}\). Let \(v_1 = te_1 - e_2\) and \(v_2 = te_2\), so that \(M_3\) is the quotient of \(S^{\oplus 2}\) by the submodule generated by \(v_1\) and \(v_2\). Now, \(\{e_1, v_1\}\) forms a basis for \(S^{\oplus 2}\), and we have \(v_2 = t^2e_1 - tv_1\). We thus find

\[
M_3 = (Se_1 \oplus Sv_1)/(Sv_1 + Sv_2) = Se_1/(St^2e_1) \cong S/(t^2).
\]

Finally, consider \(M_4\). Let \(w_1 = te_1 - e_2\) and \(w_2 = (t-1)e_2\), so that \(M_4\) is the quotient of \(S^{\oplus 2}\) by the submodule generated by \(w_1\) and \(w_2\). As before, \(\{e_1, w_1\}\) is a basis for \(S^{\oplus 2}\), and we have \(w_2 = (t-1)(te_1 - w_1)\). We thus find

\[
M_4 = (Se_1 \oplus Sw_1)/(Sw_1 + S(t-1)(te_1 - w_1)) = Se_1/(S(t-1)e_1) \cong S/(t(t-1)).
\]

We thus see that \(M_2\) and \(M_4\) are isomorphic (by the Chinese remainder theorem). All other pairs are non-isomorphic by the uniqueness part of the structure theorem. This can also be seen directly by considering annihilators: the annihilator of \(M_1\) is \((t)\), of \(M_2 \cong M_4\) is \((t(t-1))\), and of \(M_3\) is \((t^2)\).