Algebra I Exam – January 2021

Problem 1. Let V be a 2-dimensional complex vector space. What is the largest value of n for which there are vectors $v_1, \dots, v_n$ in V such that $v_1^{\otimes 3}, \dots, v_n^{\otimes 3}$ are linearly independent? Here $v^{\otimes 3}$ denotes the element $v \otimes v \otimes v$ of $V^{\otimes 3} = V \otimes V \otimes V$.

Problem 2. Let X be an $n \times n$ matrix with entries in $\mathbb{C}$. Let

$$V = \{Y \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \ : \ XY = YX\},$$

which is a vector subspace of $\mathrm{Mat}_{n\times n}(\mathbb{C})$. Show that $\dim_{\mathbb{C}} V \geq n$.

Problem 3. Let $R = \mathbb{Q}[x, y]$. Show that there are only finitely many ideals of R which contain the ideal $\langle x, y\rangle \cap \langle x - 1, y - 1\rangle$.

Problem 4. Let A be a finite abelian group such that $a^{10} = 1$ for all a in A. Suppose that A has exactly 168 elements of order 10. What is the order of A?

Problem 5. Let $S = \mathbb{Q}[t]$. We'll write elements of $S^{\oplus 2}$ as column vectors. Define the following S-modules:

$$\begin{array}{rcl} M_1 & = & S^{\oplus 2}/\left(S\left[\begin{smallmatrix} t \\ 0 \end{smallmatrix}\right] + S\left[\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right]\right) \\ M_2 & = & S^{\oplus 2}/\left(S\left[\begin{smallmatrix} t \\ 0 \end{smallmatrix}\right] + S\left[\begin{smallmatrix} 0 \\ t-1 \end{smallmatrix}\right]\right) \\ M_3 & = & S^{\oplus 2}/\left(S\left[\begin{smallmatrix} t \\ -1 \end{smallmatrix}\right] + S\left[\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right]\right) \\ M_4 & = & S^{\oplus 2}/\left(S\left[\begin{smallmatrix} t \\ -1 \end{smallmatrix}\right] + S\left[\begin{smallmatrix} 0 \\ t-1 \end{smallmatrix}\right]\right) \end{array}.$$

Two of these modules are isomorphic to each other. Prove that they are isomorphic, and show that the other pairs of modules are nonisomorphic.