ALGEBRA II

We use the following standard notation: $\mathbb{Z}$ is the ring of integers, $\mathbb{Q}$ is the field of rational numbers, $\mathbb{R}$ is the field of real numbers, $\mathbb{C}$ is the field of complex numbers, and $\mathbb{F}_q$ is the finite field with q elements (where $q = p^e$ for some prime p and $e \geq 1$).

(1) Let K be a subfield of $\mathbb{C}$ such that K is a Galois extension of $\mathbb{Q}$ with $[K : \mathbb{Q}]$ odd. Show that $K \subset \mathbb{R}$.

(2) Let p be an odd prime number. Form the semidirect product $G = \mathbb{F}_p \rtimes \mathbb{F}_p^*$ for the standard (scalar multiplication) action of $\mathbb{F}_p^*$ on $\mathbb{F}_p$. Let ℓ be a prime. Calculate the cardinality of the set of all group homomorphisms G to the cyclic group $\mathbb{Z}/\ell\mathbb{Z}$ in the following cases:

   (a) ℓ is a prime number different from p.

   (b) $\ell = p$.

(3) Let p be a prime number, and let $k = \mathbb{F}_p(x)$. For $f(x) \in k$, let $K_f = k[y]/(y^p - f(x))$. Show that the ring K_f is a field exactly when $f(x)$ is not a p-th power.

(4) Fix a prime number p. Describe a p-Sylow subgroup in each of the following groups:

   (a) $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$

   (b) $\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$

   Here we use the following notation: for any ring R, the group $\mathrm{GL}_2(R)$ is the group (2×2) invertible matrices over R (with group operation being matrix multiplication).

(5) Let L/K be an algebraic extension of fields of characteristic 0. Assume that for every $\alpha \in L$, the extension $K(\alpha)/K$ has degree ≤ 2. Show that $[L : K] \leq 2$.

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