Algebra II Exam – Fall 2020

The symbols $\mathbb{F}_p$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ denote the finite field with p elements, the integers, the rational numbers, and the complex numbers

Problem 1. Can the alternating group A_{2020} be generated by three permutations x, y, and z satisfying $xy = zxy$, $xz = zx$, and $yz = zy$? Be sure to justify your answer.

Problem 2. Let G be the group of all invertible upper-triangular 2×2 real matrices (with group law matrix multiplication). Let H be the subset of G consisting of all elements of the form g^2 with $g \in G$. Show that H is a subgroup of G and compute its index.

Problem 3. Let a and b be rational numbers such that $a^2+b^2 = 1$, and suppose that $a + bi$ is not a square in the field $\mathbb{Q}(i)$, where $i = \sqrt{-1}$. Let $K = \mathbb{Q}(i, \sqrt{a + bi})$. Show that K is Galois over $\mathbb{Q}$ and describe its Galois group.

Problem 4. Let ℓ be an odd prime number, let p be a prime congruent to 1 modulo ℓ, and let $G = \mathrm{GL}_2(\mathbb{F}_p)$. Give an example of an ℓ-Sylow subgroup of G, and compute how many ℓ-Sylow subgroups G has. You may use without proof the fact that the multiplicative group $\mathbb{F}_p^\times$ is cyclic.

Problem 5. Let p be a prime number and let K be a field of characteristic p. Let $a, b \in K$, with $a \neq 0$, and let L be the splitting field of $x^p - ax - b$ over K. Show that L/K is Galois and that its Galois group is solvable.