The symbols $\mathbb{F}_p$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ denote the finite field with $p$ elements, the integers, the rational numbers, and the complex numbers.

**Problem 1.** Let $V$ and $W$ be finite dimensional complex vector spaces, and let $v_1, v_2 \in V$ and $w_1, w_2 \in W$. Let $T$ be the tensor $v_1 \otimes w_1 + v_2 \otimes w_2$ in $V \otimes W$. Show that, if $T$ is of the form $x \otimes y$ for some $x \in V$ and $y \in W$, then either $v_1$ and $v_2$ are linearly dependent or else $w_1$ and $w_2$ are linearly dependent.

**Problem 2.** Let $I$ be the ideal $\langle x^2 + 1, y^2 + 1 \rangle$ in the ring $\mathbb{Q}[x, y]$. Show that $I$ is not prime, and give a prime ideal containing $I$. (We remind the reader that the ideal $(1)$ is not considered to be prime.)

**Problem 3.** Let $p(x, y)$ be an irreducible polynomial with complex coefficients. Let $R$ be the subring of $\mathbb{C}(x, y)$ consisting of all rational functions $f(x, y)/g(x, y)$ such that $p(x, y) \nmid g(x, y)$. Show that every ideal of $R$ is principal.

**Problem 4.** Counting up to isomorphism, how many abelian groups $G$ are there such that $G$ is generated by 3 elements and $g^4 = 1$ for all $g \in G$?

**Problem 5.** Let $A$ be a $3 \times 3$ integer matrix. Suppose that, considered as a matrix over $\mathbb{C}$, the matrix $A$ has Jordan form

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Let $p$ be a prime integer and let $\bar{A}$ be the reduction of $A$ modulo $p$. What are the possible Jordan forms of $\bar{A}$, considered as a matrix over the algebraic closure of $\mathbb{F}_p$?