Qualifying Exam Algebra May 2019

Morning

Instructions: Write your ID number in the upper right corner on each sheet that you hand in. Justify your answers.

(1) Suppose that A is a complex 7×7 matrix that satisfies the relation $A^5 = 2A^4 + A^3$. Given that the rank of A is 5 and the trace of A is 4, what is the Jordan canonical form of A?

(2) Let S be the set of all infinite sequences $(x_1, x_2, x_3, \dots)$ in $\mathbb{R}$ for which the limit $\lim_{n\to\infty} x_n$ exists. We define an addition and a multiplication on S by

$$\begin{aligned} (x_1, x_2, x_3, \dots) + (y_1, y_2, y_3, \dots) &= (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots) \\ (x_1, x_2, x_3, \dots) \cdot (y_1, y_2, y_3, \dots) &= (x_1 y_1, x_2 y_2, x_3 y_3, \dots) \end{aligned}$$

(a) Show that S is a commutative ring with identity.

(b) Let $\mathfrak{m} \subseteq S$ be the set of all $(x_1, x_2, x_3, \dots)$ with $\lim_{n\to\infty} x_n = 0$. Show that $\mathfrak{m}$ is a maximal ideal of S.

(3) Let $\alpha = \sqrt[3]{2}$ and consider the field $K = \mathbb{Q}(\alpha)$. Suppose that $\beta = p + q\alpha + r\alpha^2$. What are the trace $\mathrm{Tr}_{K/\mathbb{Q}}(\beta)$ and norm $\mathrm{N}_{K/\mathbb{Q}}(\beta)$ of β? (Your answers should be polynomials in p, q, r with coefficients in $\mathbb{Q}$.)

(4) A Hermitian complex matrix H is said to have *signature* (p, q, r) if there exists an invertible matrix P so that $P^\star H P$ is a real diagonal matrix whose diagonal has p positive entries, q negative entries and r zeroes. Here $P^\star = \overline{P}^t$ is the complex transpose matrix. Let A be an $n \times n$ Hermitian matrix. Form the $2n \times 2n$ block matrix

$$M = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$$

and let d be the nullity of A. Prove that the signature of M is $(n-d, n-d, 2d)$.

(5) Suppose that $p > q$ are prime numbers, and that G is a group of order $p^2 q^2$.

(a) Show that $p = 3$ or G has a normal subgroup of order p^2.

(b) If $p = 3$ (and therefore $q = 2$), show that G has a normal subgroup of order 3 or 9.

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Afternoon

Instructions: Write your ID number in the upper right corner on each sheet that you hand in. Justify your answers.

(1) Suppose that V is an n-dimensional K-vector space and $v \in V$ is nonzero.
(a) Show that for p with $0 \leq p < n$ there exists a unique linear map

$$\varphi_p : \bigwedge^p V \to \bigwedge^{p+1} V$$

with the property that

$$\varphi_p(w_1 \wedge w_2 \wedge \cdots \wedge w_p) = v \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_p.$$

(b) What is the rank of φ_p?

(2) Suppose that L/K is a field extension, and $A, B \in \mathrm{Mat}_{n,n}(K)$ are $n \times n$ matrices with entries in K. Suppose that there exists an invertible matrix $C \in \mathrm{Mat}_{n,n}(L)$ with $CAC^{-1} = B$. Show that there exists an invertible matrix $D \in \mathrm{Mat}_{n,n}(K)$ with $DAD^{-1} = B$. (Hint: Think about the invariant factors or the rational canonical form of the matrix A.)

(3) Let $a \in \mathbb{Q}$ and let $n \geq 2$ be an integer. Prove that the Galois group of $x^n - a$ over $\mathbb{Q}$ is solvable. Prove also that its order is at most $n(n-1)$.

(4) Suppose that $\mathcal{P}$ is a property of some groups. We say that a group is *virtually* $\mathcal{P}$ if it has a finite index subgroup which has property $\mathcal{P}$. Prove that if the group G is virtually solvable, then so is every subgroup and every quotient group of G.

(5) Let R be a commutative ring with identity and M be an R-module (not necessarily finitely generated). Suppose that $a_1, a_2, \ldots, a_n \in R$ such that $(a_1, a_2, \ldots, a_n) = R$ and $a_i a_j M = 0$ for $i \neq j$. Show that we have a direct sum decomposition

$$M = a_1 M \oplus a_2 M \oplus \cdots \oplus a_n M.$$