QR Exam Algebra
January 7, 2019
Morning

Justify your answers.

(1) Let $\mathbb{F}$ be a field and V be a finite dimensional $\mathbb{F}$-vector space with bilinear form $b : V \times V \to \mathbb{F}$. Prove that the following statements are equivalent
  (a) b is nondegenerate (i.e., if $x \in V$ and $b(x, y) = 0$ for all y, then $x = 0$).
  (b) For every subspace W of V and every linear functional $g : W \to \mathbb{F}$, there exists $v \in V$ so that $g(x) = b(x, v)$ for all $x \in W$.

(2) Suppose that K, L and M are fields with $K \subseteq L \subseteq M$ such that M/K is a Galois extension of degree 1000 and L/K is a field extension of degee 8. Show that L/K is also a Galois extension.

(3) Suppose that V is an n dimensional $\mathbb{C}$-vector space and $A : V \to V$ is a linear map.
  (a) Show that there exists a unique linear map $\varphi_A : \bigwedge^2 V \to \bigwedge^2 V$ such that

$$\varphi_A(v \wedge w) = Av \wedge w + v \wedge Aw$$

  for all $v, w \in V$.
  (b) Assume that $n = 4$ and the Jordan normal form of A is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

  What is the Jordan normal form of φ_A?

(4) Suppose that $\mathbb{F}$ is a field, V is an n-dimensional $\mathbb{F}$-vector space, $R \subseteq \operatorname{End}(V)$ is a commutative subalgebra and $v \in V$ is a vector with $Rv = \{Av \mid A \in R\} = V$. Show that $\dim R = n$.

(5) Let G be a finite solvable group.
  (a) Let K be a minimal normal subgroup of G. Prove that K is an abelian p-group for some prime number p and that K is elementary abelian ($x^p = 1$ for all $x \in K$).
  (b) Let M be a maximal subgroup of G. Prove that the index $|G : M|$ is a prime-power.

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Afternoon

Justify your answers.

(1) Suppose that H, K are subgroups of an infinite group G. Suppose that their indices $m := |G : H|$ and $n := |G : K|$ are finite. Prove that $G = HK$ if m, n are relatively prime.

(2) Suppose that R is a unique factorization domain (UFD) for which every nonzero prime ideal is maximal.
   (a) Show that every prime ideal is generated by one element.
   (b) Show that R is a principal ideal domain (PID).

(3) Let $\mathbb{F}_p$ be the field with p elements. Suppose that $a(x) \in \mathbb{F}_p[x]$ is a polynomial of degree 5. Show that $a(x)$ is irreducible if and only if $\gcd(a(x), x^{p^2} - x) = 1$.

(4) Recall that the nilpotence class of a nilpotent group is the minimum length of a central series. Suppose that G is a nilpotent group and not abelian. Let G' be the commutator subgroup and let $x \in G$. Prove that $\langle G', x \rangle$ is a proper subgroup of G. *Hint:* show that the nilpotence class of $\langle G', x \rangle$ is less than than the nilpotence class of G.

(5) Let V be an n-dimensional vector space over a subfield $\mathbb{F}$ of the reals, and let $b : V \times V \to \mathbb{F}$ be a positive definite symmetric bilinear form. An *isometry* of b is a linear transformation $T : V \to V$ which preserves b, i.e., $b(Tx, Ty) = b(x, y)$ for all $x, y \in V$. A *reflection* on V is a linear transformation of the form $r_v : x \mapsto x - 2\frac{b(x,v)}{b(v,v)}v$ for some nonzero vector $v \in V$. A reflection is an isometry of b (you do not have to verify this). Prove that T is a product of k reflections on V, for some $k \leq n$.