May 2016, Qualifying Review Algebra, Morning

Justify all of your answers. We write $\mathbf{C}$, $\mathbf{F}_p$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{Z}$ for the complex numbers, the field with p elements, the rational numbers, the real numbers and the integers respectively.

Problem 1.
(a) Suppose I is an ideal in a principal ideal domain R such that $I^2 = I$. Show that $I = (0)$ or $I = R$.
(b) Give an example of an ideal I in a commutative ring R such that $I^2 = I$ but I is not (0) or R. Justify your answer.

Problem 2. Suppose that A is an invertible symmetric $n \times n$ matrix with real entries. Show that there exist invertible real matrices R and S such that $I_n = RAR^t - SAS^t$, where I_n is the $n \times n$ identity matrix.

Problem 3. Let A be a 2×2 matrix with real entries. Suppose there exist non-zero vectors $v, w \in \mathbf{R}^2$ such that $\|A^n v\| \to 0$ as $n \to \infty$ and $\|A^n w\| \to \infty$ as $n \to \infty$, where $\|\cdot\|$ denotes the length of a vector. Show that A is diagonalizable over the reals, i.e., there exists an invertible real matrix S such that SAS^{-1} is diagonal.

Problem 4. Suppose $a, b \in \mathbf{Q}$ and $\zeta = e^{2\pi i/3}$ is a primitive third root of unity. Let $L = \mathbf{Q}(\zeta, \sqrt[3]{a}, \sqrt[3]{b})$.

(a) Show that the field extension $L/\mathbf{Q}$ is Galois.
(b) Suppose that none of the numbers a, b, ab, ab^2 is a third power of a rational number. Show that $L/\mathbf{Q}$ has degree 18.

Problem 5. Let S_3 act on $V = \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ by permuting the tensor factors. Show that there are infinitely many subspaces W of V that are stable by S_3 (that is, $gW \subset W$ for all $g \in S_3$).

May 2016, Qualifying Review Algebra, Afternoon

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Problem 1. Show that every group of order $224 = 2^5 \cdot 7$ has an element of order 14.

Problem 2. Suppose that A is a 6×6 complex matrix with minimum polynomial $x^6 + x^5 - x^4 - x^3 = x^3(x-1)(x+1)^2$. Determine the characteristic polynomial and minimal polynomial of A^2 and justify your answer.

Problem 3. Suppose A and B are invertible 2×2 complex matrices.

(a) Show that there exists a linear transformation

$$F\colon \mathbf{C}^2 \otimes \mathbf{C}^2 \to \mathbf{C}^2 \otimes \mathbf{C}^2$$

with $F(v \otimes w) = (Av) \otimes (Bw) - (Bv) \otimes (Aw)$ for all $v, w \in \mathbf{C}^2$.

(b) Show that the rank of F is at most 2.

Problem 4. Let F be the field $\mathbf{C}(x_1, \ldots, x_n)$. Let S_n act on this field by permuting the variables, and let $E = F^{S_n}$ be the fixed field. Suppose that $\Phi \in E[T]$ is a polynomial of degree at most $n-1$ such that $\Phi(x_i) = \Phi(x_j)$ for all $1 \leq i, j \leq n$. Show that Φ is constant.

Problem 5. Consider the polynomial $p(x) = x^9 + 1 \in \mathbf{F}_2[x]$.

(a) Show that $p(x)$ splits over the field $\mathbf{F}_{64}$.

(b) Show that $p(x) = (x+1)(x^2+x+1)(x^6+x^3+1)$ is the irreducible factorization of p. (It is enough to show that the three factors are irreducible, you don't have to do the multiplication!)

(c) How many units does the ring $\mathbf{F}_2[x]/(p(x))$ have? Justify your answer.